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Space-Time-Fractional Nonlinear Differential Equations in Mathematical-Physics

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Abstract

In this paper, some space-time fractional equations that can be used to describe phenomena in physics are derived. The semi-inverse method is used to evaluate the Euler-Lagrange equation in its regular form. Then the variational technique of Agrawal is used to derive the space-time fractional form of the evolution equation. This technique is applied here to derive the Burgers, Korteweg-de Vries, Kadomtsev-Petviashvili and Boussinesq equations.

Keywords: space-time fractional equations; semi-inverse method; variational technique; nonlinear evolution equations

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1. Introduction

Derivatives and integrals of fractional order have found many applications in recent studies in mechanics and physics. In a fairly short period of time the list of such applications becomes long. For example, it includes chaotic dynamics [1], mechanics of fractal media [2], quantum mechanics [3], physical kinetics [4], plasma physics [5], astrophysics [6], mechanics of non-Hamiltonian systems [7], theory of long range interaction [8], anomalous diffusion and transport theory [9] and many other physical topics [10, 11].

Traditional Lagrangian and Hamiltonian mechanics cannot be used with non-conservative forces such as friction. The most methods of classical mechanics deal with conservative systems, while

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almost all process observed in the physical real world are non-conservative. It was shown that non-integer derivatives in the Lagrangian describe non-conservative forces. Riewe [12, 13] derived a method using a fractional Lagrangian that leads to a fractional Euler-Lagrange equation that is, in some sense, equivalent to the desired equation of motion. Hamilton's equations are derived from the Lagrangian and are equivalent to the Euler-Lagrange equation. Further study of the fractional Euler-Lagrange can be found in the works of Agrawal [14, 15]. He presented generalized Euler-Lagrange equations for unconstrained and constrained fractional variational problems. Baleanu and coworkers [16, 17] used the fractional Euler-Lagrange equation to model fractional Lagrangian and Hamiltonian formulations. El-Wakil et al derived the time fractional forms of some mathematical-physics equations [18] using Agrawal's variational method and used them to describe the electrostatic integral and derivative called modified Riemann-Liouville [20, 21] and is used to solve some problems [22, 23].

In this paper, the space-time-fractional Burgers, Korteweg-de Vries (KdV), Kadomtsev-Petviashvili (KP) and Boussinesq equations are derived using Agrawal's variational technique and the modified Riemann-Liouville (mRL) derivative.

2. Derivation of Space-Time-Fractional Equations

The semi-inverse method [24] and the Agrawal's variational technique [14, 15] are applied to derive some of the evolution equations that are considered to describe physical phenomena. In this paper, space-time fractional Burgers, KdV, KP and Boussinesq equations are derived.

2.1. Space-Time-Fractional Burgers Equation

The Burgers equation has been appearing in many fields of research such as the model of turbulence, acoustic wave propagation, astrophysics and the study of surface growth. Also, it can be used for a model of bore if dispersion effect is ignored and the unidirectional wave is considered. The regular Burgers equation has the form

\[
\frac{\partial}{\partial t}v(x,t) + Av(x,t)\frac{\partial}{\partial x}v(x,t) + B\frac{\partial^2}{\partial x^2}v(x,t) = 0,
\]

where \(v(x,t)\) is a field variable, \(A\) is the nonlinear coefficient and \(B\) is the dissipation coefficient.

The space-time-fractional Burgers (STFB) equation in \((1+1)\) dimension can be formulated as follows:

Using the potential function \(U(x,t)\) where \(v(x,t) = U_x(x,t)\) gives the potential equation of the regular Burgers equation in the form

\[
U_{xt}(x,t) + AU_x(x,t)U_{xx}(x,t) + BV_{xx}(x,t) = 0,
\]

where the subscripts denote the partial differentiation of the function with respect to the parameter. The Euler-Lagrange equation of the regular Burgers equation can be derived using the semi-inverse method [24] as follows:

The functional of the potential equation can be represented by

\[
J(U) = \int_R \int_T dt U(x,t)\{c_1U_{xt}(x,t) + \frac{1}{2}c_2 A[U_x^2(x,t)]_x + c_3 Bv_{xx}(x,t)\},
\]
where \( c_1, c_2 \) and \( c_3 \) are Lagrangian multipliers. Integrating this functional by parts, applying the variation of this functional with respect to \( U(x,t) \), assuming that \( v_{xx}(x,t) \) is a fixed function, integrating by parts using \( U_t|_{t=0}=U_x|_{x=0}=0 \), optimizing this variation of functional, \( \partial J(U) = 0 \) and comparing the resultant relation with the potential equation (1.2) give the Lagrangian multipliers as: \( c_1 = \frac{1}{2}, \ c_2 = \frac{1}{3} \text{ and } c_3 = 1. \)

Substituting these multipliers into (1.3) yields directly the Lagrangian of the potential form of Burgers equation as

\[
L(U,U_t,U_{xx}) = -\frac{\alpha}{2} U_t(x,t)U_x(x,t) - \frac{1}{6} \alpha U^3_{xx}(x,t) + BU(x,t)v_{xx}(x,t). \tag{1.4}
\]

Similarly, the Lagrangian of the STFB equation could be written in the form

\[
F(U, D_x^\beta U, D_x^{\beta}U) = -\frac{\alpha}{2} D_x^\beta U(x,t)D_x^{\beta}U(x,t) - \frac{1}{6} \alpha [D_x^{\beta}U(x,t)]^3 + BU(x,t)D_x^{\beta}v(x,t). \tag{1.5}
\]

In this equation, \( D_x^\beta f(z) = D_x^\beta [D_x^{(\beta)} f(z)] \) while the fractional derivative \( D_x^{(\beta)} f(z) \) is the mRL fractional derivative defined by [20, 21]

\[
D_x^{(\beta)} f(z) = \frac{1}{\Gamma(1-\gamma)} \frac{d}{dz} \left\{ \int_a^z \frac{f(\zeta) - f(a)}{(\zeta - z)^\gamma} d\zeta \right\}, \quad 0 \leq \gamma < 1. \tag{1.6}
\]

The fractional form of the Burgers equation can be derived following a variational technique derived by Agrawal [13, 14] as follows.

The functional of the STFBurgers equation takes the form

\[
J_f(U) = \int_a^b (dx)^\alpha \int_a^b (dt)^\gamma F(U, D_x^\beta U, D_x^{\beta}U), \tag{1.7}
\]

where [20, 21]

\[
\int_a^b (d\tau)^\gamma f(\tau) = \gamma \int_a^b d\tau(t-\tau)^\gamma f(\tau). \tag{1.8}
\]

The variation of this functional with respect to \( U(x,t) \), integration by parts and optimization of this variation, \( \partial J_f(U) = 0 \), leads to the fractional Euler-Lagrange equation in the form

\[
\left( \frac{\partial F}{\partial U} \right) - D_x^\beta \left( \frac{\partial F}{\partial D_x^\beta U} \right) - D_x^{\beta} \left( \frac{\partial F}{\partial D_x^{\beta}U} \right) = 0, \tag{1.9}
\]

where the fractional integration by parts of the mRL fractional derivative is given by [20, 21]

\[
\int_a^b (dz)^\gamma f(z)D_x^{(\beta)} g(z) = \Gamma(1+\gamma)[g(z) f(z) \bigg|_a^b - \int_a^b (dz)^\gamma g(z) D_x^{(\beta)} f(z)], \quad f(z), g(z) \in [a,b]. \tag{1.10}
\]

Substituting the Lagrange of the STFB (1.5) into Euler-Lagrange formula (1.9) and using the definition \( v(x,t) = D_x^{\beta}U(x,t) \) gives

\[
D_x^{(\beta)} v(x,t) + AD_x^\beta v(x,t) + B D_x^{\beta} v(x,t) = 0, \tag{1.11}
\]

where the fractional differentiation of multiplication of two functions is given as [20, 21]

\[
D_x^{(\beta)} [f(z)g(z)] = [D_x^{(\beta)} f(z)] g(z) + f(z) [D_x^{(\beta)} g(z)]. \tag{1.12}
\]
Equation (1.11) is what is called the space-time-fractional Burgers equation.

2.2. Space-Time-Fractional KdV Equation

The KdV equation has a central place in a model for waves on shallow water and it is an example of the propagation of dispersive and nonlinear waves. The KdV first formulated as part of an analysis of shallow-water waves in canals, it has subsequently been found to be involved in a wide range of physics phenomena, especially those exhibiting shock waves, traveling waves and solitons. Certain theoretical physics phenomena in the quantum mechanics domain are explained by means of a KdV model. It is used in fluid dynamics, aerodynamics, and continuum mechanics as a model for shock wave formation, solitons, turbulence, boundary layer behavior and mass transport.

The regular KdV equation in (1+1) dimensions is given by

\[ \frac{\partial}{\partial t} \psi(x,t) + A \psi(x,t) \frac{\partial}{\partial x} \psi(x,t) + C \frac{\partial^3}{\partial x^3} \psi(x,t) = 0, \]  \( (2.1) \)

where \( \psi(x,t) \) is a field variable, \( A \) is the nonlinear coefficient and \( C \) is the dispersion coefficient.

The space-time-fractional KdV (STFKdV) equation in (1+1) dimension can be formulated as follows:

Using the potential function \( U(x,t) \) where \( \psi(x,t) = U_x(x,t) \) gives the potential equation of the regular KdV equation (2.1) in the form

\[ U_{tt}(x,t) + AU_{xx}(x,t)U_{xx}(x,t) + CU_{xxxx}(x,t) = 0, \]  \( (2.2) \)

where the subscripts denote the partial differentiation of the function with respect to the parameter.

Using the semi-inverse method [24], the functional of the potential equation can be represented by

\[ J(U) = \int_{R} dx \int_{T} dt \int_{T} dt U(x,t) [c_1 U_{xx}(x,t) + \frac{1}{2} c_2 A[U_{xx}^2(x,t)]_{x} + c_4 U_{xxxx}(x,t)], \]  \( (2.3) \)

where \( c_1, c_2 \) and \( c_4 \) are constant Lagrangian multipliers.

Integrating equation (2.3) by parts, applying the variation optimum condition \( \delta J(U) = 0 \), integrating each term by parts using the conditions \( U_{t} |_{T} = U_{x} |_{R} = U_{xx} |_{R} = 0 \) and equating the resultant equation with (2.2), so the unknown multipliers become \( c_1 = \frac{1}{2}, c_2 = \frac{1}{2} \) and \( c_4 = \frac{1}{3} \).

In addition, the functional relation yields directly the Lagrangian form of the regular KdV equation as

\[ L(U,U_t,U_{xx},U_{xxx}) = \frac{1}{2} U_{t} U_{xx} + \frac{1}{2} A[U_{xx}(x,t)]^3 - \frac{1}{2} C[U_{xxxx}(x,t)]^2. \]  \( (2.4) \)

Similarly, the Lagrangian of the space-time-fractional KdV (STFKdV) equation could be written using the JRL fractional derivative \( D^\beta_x f(z) \) defined by (1.6) in the following form

\[ F(D^\beta_x U, D^\beta_x U, D^\beta_x U) = \frac{1}{2} D^\beta_x U(x,t) D^\beta_x U(x,t) + \frac{1}{2} A[D^\beta_x U(x,t)]^3 - \frac{1}{2} C[D^\beta_x U(x,t)]^2. \]  \( (2.5) \)

The functional of the STFKdV equation will take the following form

\[ J_{F}(U) = \int_{R} (dx)^{\beta} \int_{T} (dt)^{\alpha} F(D^\beta_x U, D^\beta_x U, D^\beta_x U). \]  \( (2.6) \)

The variation of this functional with respect to \( U(x,t) \), integration by parts of this variation using (1.10) and optimizing the variation of the functional, \( \delta J_{F}(U) = 0 \), obviously leads to the Euler–Lagrange equation for the STFKdV equation in the form
Substituting the Lagrange of the STFKdV given by (2.5) into this Euler-Lagrange formula and using the definition \( v(x,t) = D_\xi^\alpha U(x,t) \) and (1.12) lead to

\[
D_\xi^\alpha v(x,t) + AD_\xi^\alpha v(x,t) + CD_\xi^\beta v(x,t) = 0.
\]  

(2.8)

This is what is called Space-Time-Fractional Korteweg-de Vries equation.

2.3. Space-Time-Fractional KP Equation

The KP equation is a nonlinear wave equation in three spatial and one temporal coordinate. It is the generalization of the KdV equation to higher dimensions. The KP equation describes the evolution of long waves of small amplitudes with weak dependence on the transverse coordinates. This equation has been found to several different problems such as the propagation of solitons in multi-component plasmas, dust acoustic waves in hot dust plasmas and dense electron-positron-ion plasma.

The regular KP equation in (3+1)-dimensions is represented in the form

\[
\frac{\partial}{\partial x} \left( v + A \frac{\partial}{\partial x} v + B \frac{\partial^3}{\partial x^3} v \right) + C \frac{\partial^2}{\partial y^2} v + E \frac{\partial^2}{\partial z^2} v = 0,
\]

(3.1)

where \( v = v(x,y,z,t) \) is a field function and \( A, B, C \) and \( E \) are constants.

The space-time fractional KP (STFKP) equation can be derived using semi-inverse method [24] and Agrawal’s variational technique [14, 15] as follows:

Defining a potential function as \( v(x,y,z,t) = U(x,y,z,t) \) gives the potential equation of the regular KP equation (3.1) by

\[
U_{xxt} + A[U_{x}]_x + BU_{xxxx} + EU_{xzz} = 0,
\]

(3.2)

where the subscripts denote the partial differentiation of the function with the parameter.

The functional of the potential equation (3.1) can be represented in the form

\[
J(U) = \int R dxdydz \int_T dt U(x,y,z,t) \{ c_1 U_{xxt} + c_2 [U_{x}]_x + c_3 BU_{xxxx} + c_4 CU_{xxy} + c_5 EU_{xxz} \},
\]

(3.3)

where \( c_1, c_2, c_3, c_4, c_5 \) are Lagrangian multipliers. Integrating this equation by parts using \( U_x \big|_{x=0} = U_x \big|_{x=1} = U_y \big|_{y=0} = U_y \big|_{y=1} = U_z \big|_{z=0} = U_z \big|_{z=1} = 0 \) where \( [U_{x}]_x \) is considered as a fixed function, taking the variation of the resultant and optimizing the variation, \( \delta J(U) = 0 \), and comparing the resultant equation with the potential function (3.2) give the Lagrangian multipliers as: \( c_1 = \frac{1}{2}, c_2 = 1, c_3 = \frac{1}{2}, c_4 = \frac{1}{2} \) and \( c_5 = \frac{1}{2} \).

Therefore, the Lagrangian of the KP equation can be represented in the form

\[
L(U_x, U_{xx}, U_{xxx}, U_{xxxx}, U_{x}, U_{x}) = -\frac{1}{2} U_{xx} U_{x} + AU_x[U_{x}]_x + \frac{1}{2} B(U_{xxx})^2 - \frac{1}{2} C(U_{x})^2 - \frac{1}{2} E(U_{x})^2.
\]

(3.4)

Similarly, the fractional Lagrangian of the KP equation can be assumed in the form

\[
F(D_\xi^\alpha U, D_\xi^\beta D_\xi^\alpha U, D_\xi^\gamma D_\xi^\beta U, D_\xi^\gamma D_\xi^\gamma D_\xi^\beta U) = -\frac{1}{2}(D_\xi^\alpha U)(D_\xi^\beta D_\xi^\alpha U) + A(D_\xi^\alpha U)[D_\xi^\beta (v D_\xi^\alpha v)]
\]

where \( \alpha, \beta, \gamma \) are constants.

\[ \]
The functional of this fractional Lagrangian is given by
\[ J_F(U) = \int_R (dx)^\beta (dy)^\beta (dz)^\beta (dt)^\alpha \ F(D^{\beta}_t U, D^{\beta}_x D^{\beta}_y U, D^{\beta}_x D^{\beta}_y U, D^{\beta}_x D^{\beta}_y U, D^{\beta}_x D^{\beta}_y U) . \tag{3.6} \]

Taking the variation of this functional, integrating this equation by parts using relation (1.11) and optimizing this relation, \( \delta J_F(U) = 0 \), Euler-Lagrange equation of STFKP equation has the form
\[ -D^{\beta}_x \left( \frac{\partial F}{\partial D^{\beta}_x D^{\beta}_y U} \right) + D^{\beta}_y \left( \frac{\partial F}{\partial D^{\beta}_y U} \right) + D^{\beta}_x \left( \frac{\partial F}{\partial D^{\beta}_x D^{\beta}_y U} \right) \]
\[ -D^{\beta}_t \left( \frac{\partial F}{\partial D^{\beta}_t D^{\beta}_y U} \right) - D^{\beta}_x \left( \frac{\partial F}{\partial D^{\beta}_x D^{\beta}_y U} \right) = 0 . \tag{3.7} \]

Substituting the fractional Lagrangian of the STFKP given by (3.5) into this Euler-Lagrange formula and applying the definition of the fractional potential function as \( D^{\beta}_t U(x, y, z, t) = v(x, y, z, t) \) lead to
\[ D^{\beta}_t D^{\beta}_x v(x, y, z, t) + A D^{\beta}_x [v(x, y, z, t) D^{\beta}_x v(x, y, z, t)] + B D^{\beta}_y D^{\beta}_x v(x, y, z, t) \]
\[ + C D^{\beta}_y v(x, y, z, t) + E D^{\beta}_x v(x, y, z, t) = 0 , \tag{3.8} \]
which is the space-time fractional Kadomtsev-Petviashvili equation.

### 2.4. Space-Time-Fractional Boussinesq Equation

The Boussinesq equation arises in hydrodynamics and some physical applications and describes propagation of waves in nonlinear and dissipative media. This equation was formulated as part of an analysis of long waves in shallow water. It was subsequently applied to problems in the percolation of water in porous subsurface strata. It also crops up in the analysis of many other physical processes.

The regular Boussinesq equation of (1+1)-dimensional has the form
\[ \frac{\partial^2}{\partial x^2} v(x, t) + A \frac{\partial^2}{\partial x^2} v^2(x, t) + B \frac{\partial^2}{\partial x^2} v(x, t) + C \frac{\partial^4}{\partial x^4} v(x, t) = 0 , \tag{4.1} \]
where \( v(x, t) \) is a field function and \( A \) is the constant nonlinear coefficient, \( B \) is the dissipation coefficient and \( C \) is the higher order dissipation coefficient.

The space-time-fractional Boussinesq equation can be formulated as follows:

Assuming a potential function \( U(x, t) \) as \( v(x, t) = U_x(x, t) \) gives the potential equation of the regular Boussinesq equation (4.1) in the form
\[ U_{xx}(x, t) + A[U(x, t)^2]_{xx} + BU_{xxx}(x, t) + CU_{xxxx}(x, t) = 0 , \tag{4.2} \]
where the subscripts denote the partial differentiation of the function with the parameter.

The functional of this equation can be represented by
\[ J(U) = \int_R \int_t dU_x(x, t)|c_1 U_{xx}(x, t)| + c_2 A[U(x, t)^2]_{xx} + c_3 BU_{xxx}(x, t) + c_4 CU_{xxxx}(x, t) , \tag{4.3} \]
where \( c_1, c_2, c_3, c_4 \) are Lagrangian multipliers.
Integrating this equation by parts where \( U_{xx} |_{x=U_x} |_{t=U_{xxx}} |_{x=R} = U_{xxxx} |_{t=R} = 0 \) and \( \{(\nu(x,t))^2\}_{XX} \) is considered as a fixed function, applying the variation of this functional with respect to \( U(x,t) \), integrating by parts, optimizing of this variation, \( \delta I(U) = 0 \) and comparing the resultant equation with (4.2) give the constant multipliers as: \( c_1 = \frac{1}{2}, c_2 = 1, c_3 = \frac{1}{2} \) and \( c_4 = \frac{1}{2} \).

The functional relation yields directly the Lagrangian of the Boussinesq equation as

\[
L(U_{xx}, U_{xxx}, U_{xxxx}) = -\frac{1}{2} \{U_{xx}(x,t)\}^2 + A[\{(\nu(x,t))^2\}_{xx} U_{3}(x,t) - \frac{1}{2} B(U_{xxx}(x,t))^2 + \frac{1}{2} C[U_{xxxx}(x,t)]^2. \tag{4.4}
\]

Similarly, the Lagrangian of the space-time-fractional version of the Boussinesq equation could be written in the form

\[
F(D^\alpha_x D^\beta_t U, D^\delta_x U, D^{\delta\alpha}_t U, D^{\delta\beta}_t U) = -\frac{1}{2} [D^\alpha_x D^\beta_t U(x,t)]^2 + A[D^{\delta\alpha}_t [\nu(x,t)]]^2 D^\beta_x U(x,t)
- \frac{1}{2} B[D^{\delta\beta}_t U(x,t)]^2 + \frac{1}{2} C[D_C^{\delta\beta\beta\beta\beta} U(x,t)]^2, \tag{4.5}
\]

where \( D^\alpha_x f = D_x^n f \) and \( D^\alpha_t f(z) \) is the mRL fractional derivative defined by (1.6).

The functional relation of the space-time-fractional Boussinesq equation takes the form

\[
J_F(U) = \int_{t} (dx)^\beta \int_{\tau} (dt)^\alpha F(D^\alpha_x D^\beta_t U, D^\delta_x U, D^{\delta\alpha}_t U, D^{\delta\beta}_t U) \cdot \tag{4.6}
\]

Integrating by parts using relation (1.10) and optimizing the variation of this functional, \( \delta J_F(U) = 0 \), the Euler-Lagrange equation of space-time-fractional Boussinesq equation has the form

\[
-D^\alpha_x \left( \frac{\partial F}{\partial D^\alpha_x D^\beta_t U} \right) + \frac{\partial F}{\partial D^\beta_t U} - D^\beta_t \left( \frac{\partial F}{\partial D^\delta_x U} \right) + D^{\delta\alpha}_t \left( \frac{\partial F}{\partial D^{\delta\alpha}_t U} \right) = 0. \tag{4.7}
\]

Substituting the Lagrange of the STFBo given by (4.5) into this Euler-Lagrange formula and using the definition of the fractional potential function as \( D^\beta_x U(x,t) = \nu(x,t) \) lead to

\[
D^\alpha_x \nu(x,t) + AD^\delta_x [\nu(x,t)]^3 + BD^{\delta\alpha}_t \nu(x,t) + CD^{\delta\beta}_t \nu(x,t) = 0, \tag{4.8}
\]

which is the space-time fractional Boussinesq equation.

3. Conclusion

Nearly, all forces in nature are non-conservative: dissipative and/or dispersive forces. The classical mechanics, using integer differential equations, treated conservative forces while the non-integer differential equations can be used to describe the non-conservative forces. Therefore, the fractional calculus has many applications through last decades.

This work tries to derive some fractional nonlinear evolution equations that have many different applications in science and engineering. The semi-inverse method [24] is applied to find the regular Lagrangian of the evolution equation and then take the similar fractional Lagrangian of that equation. The variational technique derived by Agrawal [14, 15] then used to calculate the Euler-Lagrange equation and then the fractional form of the evolution equation. This technique is considered in this work to derive for example three of the most famous evolution equations that are appeared in many physical and engineering problems. They are named Burgers, Kortweg-de Vries, Kadomtsev-Petviashvili and Boussinesq equations.
Many different techniques that are used to solve the nonlinear evolution equations are modified to solve the fractional differential equations. Solutions of the fractional nonlinear differential equations and applications to describe physical models are plans to other future works.

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References


New Application of the Fractional Sub-equation Method

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Abstract

In this letter, the fractional Riccati equation with Riemann-Liouville derivatives has been successively used to find the explicit solutions of the space-time derivative of order $\alpha$, $0 < \alpha < 1$ are considered of coupled coupled KdV equations with variable coefficients. The proposed method can also be applied to other coupled nonlinear fractional differential equations with variable coefficients arising in mathematical physics. As a results, three types of exact solutions can be diretly evaluated.

Keywords: Riemann-Liouville derivative, Fractional sub-equation method, Coupled KdV equations with variable coefficients, Exact solutions

1. Introduction

The nonlinear fractional differential equations in mathematical physics are playing a major role in various fields. These equations appear in a wide great array discretization span, Jumarie’s defined the fractional derivative in of contexts, such as physics, biology, engineering, signal processing, systems identification, control theory, finance and fractional dynamics \cite{1-3}. Finding approximate and exact solutions to fractional differential equations is an important task. Consequently, a large amount of literatures developed concerning the solution of fractional differential equations in nonlinear dynamics \cite{4-9}.

During the past decades, both mathematicians and physicists have devoted considerable effort to the study of explicit solutions to nonlinear integer-order differential equations. Many powerful methods have been presented \cite{10-28}. Among them, the variational iteration method \cite{17-19}, the A domain decomposition method \cite{20-22} and the homotopy perturbation method \cite{23}, are relatively new

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approaches providing an analytical approximation to linear and nonlinear problems, and they are particularly valuable as tools for scientists and applied mathematicians, because they provide immediate and visible symbolic terms of analytic solutions, as well as a numerical approximate solution to both linear and nonlinear differential equations without linearization or discretization.

This paper is organized as follows: In section 2, basic definitions of Jumarie's Riemann-Liouville derivative and the fractional sub-equation method are given. In section 3, we consider the method in detail with the space time fractional coupled KdV equations with variable coefficients. Some conclusion and discussions are shown in section 4.

2. Jumarie's Riemann-Liouville derivative and fractional sub-equation method

In what follows, we summarize of the proposed method as presented in [25–27]. To investigate the behavior of fractional models, several versions of fractional derivatives have been proposed, i.e., the Jumarie's Riemann-Liouville derivative of order $\alpha$ is defined as [8,9]

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_0^t (t-x)^{-\alpha} (f(\xi) - f(0))d\xi, \quad 0 < \alpha < 1$$ (1)

$$D_x^\alpha f(t) = (f'^\alpha(x))^{\alpha-n}, \quad n < \alpha < n+1, \quad n > 1$$ (2)

where $f : R \rightarrow R$, $x \rightarrow f(x)$ denotes a continuous (but not necessarily first order differentiable) function. We can obtain the following properties.

The properties of the fractional Riemann-Liouville are summarized as

$$D_x^{\gamma} x^\gamma = \frac{\Gamma(1+\gamma)}{\Gamma(1+\gamma-\alpha)}, \quad \gamma > 0,$$ (3)

$$D_x^\alpha (u(x)v(x)) = v(x)D_x^\alpha u(x) + u(x)D_x^\alpha v(x),$$ (4)

$$D_x^\alpha [f(u(x))] = f_x^\alpha D_x^\alpha u(x),$$

$$D_x^\alpha [f(u(x))] = f_x^\alpha f(u(u_x)^\alpha),$$ (5)

which are direct consequences of the equality $d^\alpha x(t) = \Gamma(1+\alpha)d\sigma(t)$ which holds from non-differentiable functions
Step 1: For a given a fractional nonlinear equation with independent variable \( x \) and \( t \) as

\[
F(u, u_x, u_{xx}, D_\alpha^u, D_\alpha^{u_x}, \ldots) = 0, \quad 0 < \alpha < 1
\]  
(6)

where \( D_\alpha^u \) and \( D_\alpha^{u_x} \) are Jumarie's Riemann-Liouville of \( u \), \( u = u(x,t) \) is an unknown function, \( F \) is a polynomial in \( u \) and its derivatives. Making use the traveling wave transformations

\[
u(x,t) = u(\xi), \quad \xi = x + ct,
\]  
(7)

where \( c \) is a constant to be determined later. Then Eq.(6) reduces to fractional ordinary differential equation as

\[
N(u, cu_x, c^\alpha D_\alpha^u, D_\alpha^{u_x}, \ldots) = 0,
\]  
(8)

Step 2: The next crucial step is that solution we are looking for is expressed in the general form

\[
u(\xi) = \sum_{j=-M}^{M} a_j \phi^j(\xi),
\]  
(9)

where \( a_j (j = 0, 1, \ldots, M) \) are constants to be determined later by balancing the linear term of the highest order derivative with nonlinear term in Eq.(8) and \( \phi(\xi) \) satisfies the fractional Riccati equation \([25]\)

\[
D_\alpha^\xi \psi(\xi) = \sigma + \psi^2(\xi), \quad 0 < \alpha < 1
\]  
(10)

In view of Zhang et al.\([25]\) derived the exact solutions to Eq.(10) as

(a) If \( \sigma < 0 \)

\[
\phi(\xi) = -\sqrt{-\sigma} \tanh_{\sqrt{-\sigma}}[\sqrt{-\sigma} \xi],
\]

(b) If \( \sigma > 0 \)

\[
\phi(\xi) = \sqrt{\sigma} \tanh_{\sqrt{\sigma}}[\sqrt{\sigma} \xi],
\]  
(11)
\[ \varphi(\xi) = -\sqrt{\sigma} \cot_{\alpha} [\sqrt{\sigma} \xi] \] 

(c) If \( \sigma = 0 \)

\[ \varphi(\xi) = -\frac{\Gamma(1+\alpha)}{\xi^\alpha + w}, w = 0 \] 

where the generalized hyperbolic and trigonometric functions are

\[
\begin{align*}
\sin_{\alpha}(\xi) &= \frac{E_{\alpha}(i\xi^\alpha) - E_{\alpha}(-i\xi^\alpha)}{2i}, \\
\cos_{\alpha}(\xi) &= \frac{E_{\alpha}(i\xi^\alpha) + E_{\alpha}(-i\xi^\alpha)}{2i}, \\
\tan_{\alpha}(\xi) &= \frac{E_{\alpha}(i\xi^\alpha) - E_{\alpha}(-i\xi^\alpha)}{E_{\alpha}(i\xi^\alpha) + E_{\alpha}(-i\xi^\alpha)}, \\
\cot_{\alpha}(\xi) &= \frac{\cos_{\alpha}(\xi)}{\sin_{\alpha}(\xi)}, \\
\tanh_{\alpha}(\xi) &= \frac{\sinh_{\alpha}(\xi)}{\cosh_{\alpha}(\xi)}, \\
\coth_{\alpha}(\xi) &= \frac{\cosh_{\alpha}(\xi)}{\sinh_{\alpha}(\xi)}.
\end{align*}
\]

\[
\begin{align*}
cosh_{\alpha}(\xi) &= \frac{E_{\alpha}(\xi^\alpha) + E_{\alpha}(-\xi^\alpha)}{2}, \\
\sinh_{\alpha}(\xi) &= \frac{E_{\alpha}(i\xi^\alpha) - E_{\alpha}(-i\xi^\alpha)}{2}
\end{align*}
\]

where \( E_{\alpha}(\xi) = \sum_{k=0}^{\infty} \frac{\xi^k}{\Gamma(1+k\alpha)} \) is the Mittag-Leffler function.

**Step 3:** Substituting Eq. (9) with (10) into (8) and setting the coefficients of the powers of \( \psi'(\xi) \) to be zero, we obtain an over-determined nonlinear algebraic system in \( a_j (j = 0, 1, 2, \ldots, M) \).

**Step 4:** Solving the nonlinear algebraic system yields the explicit expressing of the parameters \( a_j (j = 0, 1, 2, \ldots, M) \). Then substituting these values into (10) we obtain the exact solutions of the nonlinear fractional Eq. (7).

3. New Application of the proposed method

To demonstrate the proposed method on nonlinear evolution equations with variable coefficients, let us consider the fractional coupled KdV equations as [28]

\[
D_i^\alpha u + d(t)uD_i^\alpha u + \beta(t)vD_i^\alpha v + \gamma(t)D_i^{3\alpha} u = 0,
\]

\[
D_i^\alpha v + \delta(t)uD_i^\alpha v + \gamma(t)D_i^{3\alpha} v = 0,
\]
where \( 0 < \alpha < 1 \), \( u \), \( v \) are functions, \( d(t), \beta(t), \gamma(t) \) and \( \delta(t) \) are all functions of variable \( t \) only, and assume that they satisfy the following conditions

\[
\beta(t) \neq 0, \quad \delta(t) \neq 0, \quad \delta(t) - \delta(t - \tau(t)) = \sigma \beta(t), \quad \gamma(t) = k \delta(t),
\]  

(17)

where \( \sigma \) and \( k \) are constants. Eqs.(15) and (16) as a simple generalized of Hirota Satsuma coupled KdV equations are physically importance. To look for the travelling wave solutions of Eqs.(15) and (16), we use the transformation

\[
u = u(\xi), \quad \chi = v(\xi), \quad \xi = kx + \int \tau(t) dt + \mu_0,
\]  

(18)

where \( \mu_0 \) and \( k \) are an arbitrary constant. Substituting Eq.(18) into (15) and (16) admits to the fractional differential equations as

\[
(\tau(t))^\alpha D_\xi^\alpha u + k^\alpha d(t) u D_\xi^\alpha u + \beta(t) k^\alpha v D_\xi^\alpha v + \gamma(t) k^{3\alpha} D_\xi^\alpha u = 0,
\]  

(19)

\[
(\tau(t))^\alpha D_\xi^\alpha v + k^\alpha \delta(t) u D_\xi^\alpha v + \gamma(t) k^{3\alpha} D_\xi^\alpha v = 0
\]  

(20)

By virtue of the technique, we suppose that the solutions can be expressed by

\[
u(\xi) = \sum_{j=-N}^{N} a_j \phi(\xi)^j,
\]

(21)

\[
u(\xi) = \sum_{j=-M}^{M} b_j \phi(\xi)^j
\]

Balancing \( D_\xi^\alpha u \), \( u D_\xi^\alpha u \) and \( v D_\xi^\alpha v \), we have \( N = M = 2 \). Substituting Eqs.(21) with the aid of Eq.(10) into Eqs.(19) and (20) and setting the coefficients of the power of \( \phi(\xi) \) to be zero, we obtain an over-determined nonlinear algebraic system in \( a_j (j = 0, 1, 2 \ldots, N) \), \( \beta(t) \), \( d(t) \), \( \gamma(t) \), \( \delta(t) \) and \( \tau(t) \). Solving the nonlinear algebraic system yields the explicit expressing of the parameters, are

**Case(1)**

\[
d(t) = d(t), \quad \delta(t) = \delta(t), \quad b_0(t) = b_0(t), \quad a_0(t) = 0, \quad b_1(t) = b_1(t), \quad a_1(t) = 0, \quad b_2(t) = b_2(t), \quad a_2(t) = b_2(t) \sigma^2,
\]
\[ b_2(t) = b_2(t), a_{-2}(t) = -12 \frac{k^{2\alpha}\gamma(t)\sigma^2}{\delta(t)}, a_2(t) = -12 \frac{k^{2\alpha}\gamma(t)}{\delta(t)}, a_0(t) = -12 \frac{\gamma(t)k^{2\alpha}b_0(t)}{\delta(t)b_2(t)}, \]

\[ \tau(t) = 4^{\frac{1}{2}} \left[ \frac{k^{3\alpha}\gamma(t)(-2\sigma b_2(t) + 3b_0(t))}{b_2(t)} \right]^{\frac{1}{2}}, \beta(t) = 144 \frac{k^{4\alpha}\gamma(t)^2(\delta(t) - d(t))}{\delta'(t)b_2'(t)} \] (22)

**Case(2)**

\[ d(t) = -12 \frac{k^{2\alpha}\gamma(t)\sigma^2}{a_{-2}(t)}, \delta(t) = -6 \frac{k^{2\alpha}\gamma(t)\sigma^2}{a_{-2}(t)}, b_0(t) = 0, a_1(t) = 0, b_{-1}(t) = -\sigma b_1(t), b_1(t) = b_1(t), \]

\[ a_{-1}(t) = 0, b_{-2}(t) = b_2(t) = 0, a_{-2}(t) = a_{-2}(t), a_2(t) = \frac{a_2(t)}{\sigma^2}, a_0(t) = a_0(t), \]

\[ \tau(t) = 2^{\frac{1}{2}} \left[ k^{3\alpha}\gamma(t)(\sigma(-2a_2(t) + 3\sigma a_0(t))) \right]^{\frac{1}{2}}, \beta(t) = 12 \frac{k^{2\alpha}\gamma(t)(\sigma a_0(t) - 2a_{-2}(t))}{\sigma b_2'(t)} \] (23)

In view of case(1), we obtain new type of exact solutions of Eqs.(15) and (16) as follows:

**Family(1)**

\[ u_{ia}(\xi) = -12 \frac{\gamma(t)k^{2\alpha}b_0(t)}{\delta(t)b_2(t)} - 12 \frac{k^{2\alpha}\gamma(t)}{\delta(t)} [-\sqrt{\sigma} \tanh_a[\sqrt{-\sigma} \xi]]^2 \]
\[ -12 \frac{k^{2\alpha}\gamma(t)\sigma^2}{\delta(t)} [-\sqrt{\sigma} \tanh_a[\sqrt{-\sigma} \xi]]^2, \]

\[ v_{ib}(\xi) = b_0(t) + b_2(t)[-\sqrt{\sigma} \tanh_a[\sqrt{-\sigma} \xi]]^2 + b_2(t)\sigma^2[-\sqrt{\sigma} \tanh_a[\sqrt{-\sigma} \xi]]^2, \]

\[ v_{ic}(\xi) = -12 \frac{\gamma(t)k^{2\alpha}b_0(t)}{\delta(t)b_2(t)} - 12 \frac{k^{2\alpha}\gamma(t)}{\delta(t)} [-\sqrt{\sigma} \coth_a[\sqrt{-\sigma} \xi]]^2 \]
\[ -12 \frac{k^{2\alpha}\gamma(t)\sigma^2}{\delta(t)} [-\sqrt{\sigma} \coth_a[\sqrt{-\sigma} \xi]]^2, \]
\[ v_{id}(\xi) = b_0(t) + b_2(t)[ -\sqrt{\sigma} \coth_{\alpha} [\sqrt{\sigma} \xi] ]^2 + b_2(t)\sigma^2[ -\sqrt{\sigma} \coth_{\alpha} [\sqrt{\sigma} \xi] ]^{-2}, \quad (25) \]

where \( \sigma < 0 \) and \( \xi = kx + \left[ 4^{\frac{1}{n}} \left[ \frac{k^{\frac{1}{\gamma}} x^{\frac{2m}{\gamma}} - 2mn_{b_0(t)} + 3n_{b_0(t)}}{b_1(t)} \right] \right] dt + \mu_0 \).

**Family(2)**

\[ u_{2a}(\xi) = -12 \frac{\gamma(t)b_{2a}(t)}{\delta(t)b_2(t)} - 12 \frac{k^{2\alpha} \gamma(t)}{\delta(t)} [\sqrt{\sigma} \tan_a [\sqrt{\sigma} \xi]]^2 - 12 \frac{k^{2\alpha} \gamma(t)\sigma^2}{\delta(t)} [\sqrt{\sigma} \tan_a [\sqrt{\sigma} \xi]]^{-2}, \]

\[ v_{2b}(\xi) = b_0(t) + b_2(t)[\sqrt{\sigma} \tan_a [\sqrt{\sigma} \xi]]^2 + b_2(t)\sigma^2[\sqrt{\sigma} \tan_a [\sqrt{\sigma} \xi]]^{-2}, \quad (26) \]

\[ u_{2a}(\xi) = -12 \frac{\gamma(t)b_{2a}(t)}{\delta(t)b_2(t)} - 12 \frac{k^{2\alpha} \gamma(t)}{\delta(t)} [\sqrt{\sigma} \cot_a [\sqrt{\sigma} \xi]]^2 - 12 \frac{k^{2\alpha} \gamma(t)\sigma^2}{\delta(t)} [\sqrt{\sigma} \cot_a [\sqrt{\sigma} \xi]]^{-2}, \]

\[ v_{2a}(\xi) = b_0(t) + b_2(t)[\sqrt{\sigma} \cot_a [\sqrt{\sigma} \xi]]^2 + b_2(t)\sigma^2[\sqrt{\sigma} \cot_a [\sqrt{\sigma} \xi]]^{-2}, \quad (27) \]

where \( \sigma > 0 \) and \( \xi = kx + \left[ 4^{\frac{1}{n}} \left[ \frac{k^{\frac{1}{\gamma}} x^{\frac{2m}{\gamma}} - 2mn_{b_0(t)} + 3n_{b_0(t)}}{b_1(t)} \right] \right] dt + \mu_0 \).

**Family(3)**

\[ u_{3a}(\xi) = -12 \frac{\gamma(t)b_{3a}(t)}{\delta(t)b_2(t)} - 12 \frac{k^{2\alpha} \gamma(t)}{\delta(t)} \left[ \frac{\Gamma(1+\alpha)}{\xi^{\alpha} + w} \right]^2 - 12 \frac{k^{2\alpha} \gamma(t)\sigma^2}{\delta(t)} \left[ -\frac{\Gamma(1+\alpha)}{\xi^{\alpha} + w} \right]^{-2}, \]

\[ v_{3b}(\xi) = b_0(t) + b_2(t)[\frac{\Gamma(1+\alpha)}{\xi^{\alpha} + w}]^2 + b_2(t)\sigma^2[\frac{\Gamma(1+\alpha)}{\xi^{\alpha} + w}]^{-2}, \quad (28) \]

where \( \sigma = 0 \) and \( \xi = kx + \left[ 4^{\frac{1}{n}} \left[ \frac{k^{\frac{1}{\gamma}} x^{\frac{2m}{\gamma}} - 2mn_{b_0(t)} + 3n_{b_0(t)}}{b_1(t)} \right] \right] dt + \mu_0 \). With the aid case (2), we can be directly evaluated a new exact solutions of Eqs.(15) and (16). For simplify should be omitted here.

**4. Conclusion**

Here, the fractional sub equation method with the Riemann-Liouville derivatives is used for constructing new exact solutions of the space-time of nonlinear fractional differential equations with variable coefficients of order \( \alpha \), \( 0 < \alpha < 1 \), namely, the coupled KdV equations. The obtained travelling wave solutions are expressed by hyperbolic, triangular and rational solutions. The solutions obtained via the
propose method have many potential applications in physics.

We point out that, the considered method provides a very effective, convenient and powerful mathematical tool for solving other nonlinear fractional space-time differential equations with variable coefficients arising in mathematical physics. This our task in future. AS a limiting case, when $\alpha = 1$ the three families of solutions back to those obtained from the classical version of coupled KdV equations (15) and (16).

References

Exact Solutions of Nonlinear Rosenau-RLW Equation

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Abstract: In this paper, exp-function method has been used to find soliton solutions of Rosenau-RLW Equation. The solution procedure of this method, by the help of symbolic computation of Maple, is of utter simplicity. The exp-function method is a powerful and straight forward mathematical tool to solve the nonlinear equations in mathematical physics.

Keywords: Rosenau-RLW Equation, Exp-function method, Soliton solutions.

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1.Introduction

Nonlinear problems are taking some important and new dimensions in the recent past. It is an established fact that most of the physical problems are nonlinear in nature and hence their appropriate solutions are of utmost importance. In the similar context, number of analytical and numerical techniques including tanh-sech [1-3], extended tanh [4-5], modified tanh [6], hyperbolic function [7], sine-cosine [8], Jacobi elliptic function expansion [9], and the first integer methods [10] has been developed Recently, He et. al. [11-13] proposed a straightforward and concise method, called exp-function method, to obtain generalized solitary solutions and periodic solutions. The Exp-function method proved to be very powerful and efficient for the solutions of wide range of nonlinear problems of diversified physical nature. Mohyud-Din [14-17] extended the same for nonlinear physical problems including higher-order BVPs; Oziz [19] tried this novel approach for Fisher’s equation; Wu et. al. [20, 21] for the extension of solitary, periodic and compacton-like solutions; Yusufoglu [22] for MBBN equations, Zhang [20] for high-dimensional nonlinear evolutions; Zhu [23, 24] for the Hybrid-Lattice system and discrete m KdV lattice; Kudryashov [25] for exact soliton solutions of the generalized evolution equation of wave dynamics; Momani [26] for an explicit and numerical solutions of the fractional KdV equation; It is to be highlighted that Ebaid [27] proved that \( c = d \) and \( p = q \) are the only relations that can be obtained by applying exp-function method to any nonlinear ordinary differential equation. The solution procedure of this method, with the aid of Maple, is of utter simplicity and this method can easily be extended to other kinds of nonlinear evolution.

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equations. Inspired and motivated by the ongoing research in this area, we use the exp-function method to obtain new solitary wave solutions for the nonlinear Rosenau-RLW Equation [28, 29].

**Theorem 1.1** [27] Suppose that \( u^{(r)} \) and \( u^{(r)} \) are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where \( r \) and \( \gamma \) are both positive integers. Then the balancing procedure using the Exp-function ansatz;

\[
U(\eta) = \sum_{n=-c}^{d} a_n \exp(n \eta) \sum_{m=-p}^{q} b_m \exp(m \eta),
\]

leads to \( c = d \) and \( p = q, \forall r, s, \Omega, \lambda \geq 1 \).

**Theorem 1.2** [27] Suppose that \( u^{(r)} \) and \( u^{(r)} \) are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where \( r, s \) and \( \Omega \) are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to \( c = d \) and \( p = q, \forall r, s, k \geq 1 \).

**Theorem 1.3** [27] Suppose that \( u^{(r)} \) and \( (u^{(r)})^{(s)} \) are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where \( r, s, \Omega \) and \( \lambda \) are all positive integers. Then the balancing procedure using the Exp-function ansatz leads to \( c = d \) and \( p = q, \forall r, s, \Omega \geq 2 \).

**Theorem 1.4** [27] Suppose that \( u^{(r)} \) and \( (u^{(r)})^{(s)} \) are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where \( r, s, \Omega, \lambda \) are both positive integers. Then the balancing procedure using the Exp-function ansatz leads to \( c = d \) and \( p = q, \forall r, s, \Omega, \lambda \geq 1 \).

### 2. Exp-function Method [11-13]

Consider the general nonlinear partial differential equation of the type

\[
P(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0.
\]

Using a transformation

\[
\eta = kx - \omega t,
\]

where \( k \) and \( \omega \) are constants, we can rewrite equation (1) in the following nonlinear ODE,

\[
Q(u, u_t, u_x, u_{xx}, u_{xxx}, \ldots) = 0.
\]

where the prime denotes derivative with respect to \( \eta \).

According to the exp-function method, which was developed by He and Wu [11], we assume that the wave solutions can be expressed in the following form

\[
u(\eta) = \sum_{n=-c}^{d} \frac{a_n \exp(n \eta)}{\sum_{m=-p}^{q} b_m \exp(m \eta)}
\]

where \( p, q, c \) and \( d \) are positive integers which are known to be further determined, \( a_n \) and \( b_m \) are unknown constants. We can rewrite equation (4) in the following equivalent form

\[
u(\eta) = \frac{a_c \exp(c \eta) + \ldots + a_d \exp(d \eta)}{b_p \exp(p \eta) + \ldots + b_q \exp(q \eta)}
\]

To determine the value of \( c, p, d \) and \( q \), we use [25].

### 3. Solution Procedure
Consider the following homogeneous Rosenau-RLW Equation
\[ u_t + u_{xxxx} - u_{xxt} + u_x + uu_x = 0, (x,t) \in [0,1] \times (0,1). \] (6)
with initial condition
\[ u(x,0) = x^2(x - 1)^2, x \in [0,1], \]
and boundary conditions
\[ u(0,t) = u(1,t) = u_t(0,t) = u_t(1,t) = 0, x \in (0,1). \]
Using (2) equation (6) can be converted to an ordinary differential equation
\[-\omega u'' - \alpha k^4 u^{(v)} + \alpha k^2 u''' + ku' + ku'' = 0, \] (7)
where the prime denotes the derivative with respect to \( \eta \). The solution of the equation (6) can be expressed in the form, equation (5). To determine the value of \( c \) and \( p \), by using [25],
\[ p = c, q = d. \] (8)

**Case 4.1.1.** We can freely choose the values of \( c \) and \( d \), but we will illustrate that the final solution does not strongly depend upon the choice of values of \( c \) and \( d \). For simplicity, we set \( p = c = 1 \) and \( q = d = 1 \) equation (5) reduces to
\[ u(\eta) = a_1 \exp[\eta] + a_0 + a_{-1} \exp[-\eta]. \] (9)
Substituting equation (9) into equation (7), we have
\[ \frac{1}{A} \left[ c_4 \exp(4\eta) + c_3 \exp(3\eta) + c_2 \exp(2\eta) + c_1 \exp(\eta) + c_0 + c_{-1} \exp(-\eta) + c_{-2} \exp(-2\eta) \right] = 0, \] (10)
where \( A = (b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta))^4 \), \( c_i \) are constants obtained by Maple 17. Equating the coefficients of \( \exp(\eta) \) to be zero, we obtain
\[ \{ c_4 = 0, c_{-3} = 0, c_{-2} = 0, c_{-1} = 0, c_0 = 0, c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0 \}. \] (11)
Solution of (11) we have four solution set satisfy the given equation (6)

**1st Solution set**
\[ \left\{ k = k, a_{-1} = 0, a_0 = \frac{b_0 a_1}{b_1}, a_1 = a_1, b_{-1} = 0, b_0 = b_0, b_1 = b_1 \right\}. \]

We, therefore, obtained the following generalized solitary solution \( u(x,t) \) of equation (6)
\[ \left\{ \begin{array}{l} \frac{b_0 a_1}{b_1} + a_1 e^{k \eta - t \omega} \\ \frac{b_0 + b_1 e^{k \eta - t \omega}}{b_1} \end{array} \right\}, \]
Figs 1 Soliton solutions of equation (6).

2nd Solution set

\[ \begin{align*}
  k &= k, a_{-1} = a_{-1}, a_0 = \frac{a_{-1}b_0}{b_{-1}}, a_1 = \frac{a_{-1}b_1}{b_{-1}}, b_{-1} = b_{-1}, b_0 = b_0, b_1 = b_1
\end{align*} \]

We, therefore, obtained the following generalized solitary solution \( u(x, t) \) of equation (6)

\[ \begin{align*}
  u(x, t) &= \left( a_{-1}e^{-kx+\tau t} + \frac{a_{-1}b_0}{b_{-1}} + \frac{a_{-1}b_1e^{kx-\tau t}}{b_{-1}} \right) \\
  &\quad \div \left( b_{-1}e^{-kx+\tau t} + b_0 + b_1e^{kx-\tau t} \right)
\end{align*} \]
Figs 2 Soliton solutions of equation (6).

3rd Solution set

\[ \{ k = k, a_{-1} = \frac{a_0 b_{-1}}{b_0}, a_0 = a_0, a_1 = 0, b_{-1} = b_{-1}, b_0 = b_0, b_1 = 0 \} \]

We, therefore, obtained the following generalized solitary solution \( u(x, t) \) of equation (6)

\[
\begin{pmatrix}
\frac{a_0 b_{-1} e^{-k x + t c}}{b_0} + a_0 \\
\frac{b_0 + b_{-1} e^{-k x + t c}}{b_0 + b_{-1} e^{-k x + t c}}
\end{pmatrix}
\]
4th Solution set

\[ k = k, a_{-1} = \frac{a_1 b_{-1}}{b_1}, a_0 = 0, a_1 = a_{1}, b_{-1} = b_{-1}, b_0 = 0, b_1 = b_1 \]

We, therefore, obtained the following generalized solitary solution \( u(x,t) \) of equation (6)

\[
\begin{bmatrix}
\frac{a_{-1} e^{-kx+\omega t}}{b_1} + a_1 e^{kx-\omega t} \\
\frac{b_{-1} e^{-kx+\omega t}}{b_1} + b_1 e^{kx-\omega t}
\end{bmatrix}
\]
Case 4.1.II. If $c = 2$ and $d = 1$ then trial solution, equation (5) reduces to
\[
\eta = \frac{a_2 \exp(2\eta) + a_1 \exp(\eta) + a_0 + a_{-1} \exp(-\eta)}{b_2 \exp(2\eta) + b_1 \exp(\eta) + b_0 + b_{-1} \exp(-\eta)}.
\] (12)
Proceeding as before, we obtain
1st solution set:
\[
\left\{ \begin{array}{l}
\omega = \omega, a_{-1} = 0, a_2 = \frac{a_1 b_2}{b_1}, a_0 = \frac{a_1 b_0}{b_1}, a_1 = a_1, b_0 = b_0, b_{-1} = 0, b_1 = b_1, b_2 = b_2
\end{array} \right\},
\]
Hence we get the generalized solitary wave solution of equation (6)
Figs 5 Soliton solutions of equation (6).

2\textsuperscript{nd} solution set:

\[
\begin{align*}
\omega &= \omega, a_0 = a_0, a_1 = \frac{a_0 b_1}{b_0}, a_2 = \frac{a_0 b_2}{b_0}, b_1 = 0, b_0 = b_0, b_1 = b_1, b_2 = b_2
\end{align*}
\]

We, therefore, obtained the following generalized solitary solution \( u(x,t) \) of equation (6)

\[
\begin{align*}
\left\{ \frac{a_0 + a_0 b_1 e^\eta}{b_0} + \frac{a_0 b_2 e^{2\eta}}{b_0} \middle| \frac{b_0 + b_1 e^\eta + b_2 e^{2\eta}}{b_0} \right\}
\end{align*}
\]
Figures 6 Soliton solutions of equation (6).

3rd solution:
\[
\left\{ \omega = \omega, a_{-1} = 0, a_0 = 0, a_1 = \frac{b_1(k^4 \omega - k^2 \omega + k - \omega)}{k}, a_2 = a_2, b_{-1} = 0, b_0 = 0, b_1 = b_1, b_2 = 0 \right\}
\]

We, therefore, obtained the following generalized solitary solution \( u(x,t) \) of equation (6)
\[
\left\{ \begin{align*}
- \frac{b_1(k^2 \omega - k^2 \omega + k - \omega)e^{bx-2\omega t}}{k} + a_2 e^{2bx-2\omega t} \\
b_1 e^{bx-\omega t}
\end{align*} \right\}
\]
In both cases, for different choices of $c, p, d$ and $q$ we get the same soliton solutions which clearly illustrate that final solution does not strongly depend upon these parameters.

4. Conclusion

Exp-function method is successfully applied to construct generalized solitary solutions of the nonlinear Rosenau-RLW Equation. It is observed that the exp-function method is very convenient to implement, use friendly and is very useful for finding solutions of a wide class of nonlinear problems.

References


Approximate approach to degradation evolution equation of cellulose

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Abstract

The Ekenstam equation is solved approximately, and the kinetics of cellulose degradation is revealed, degree of polymerization (DP) changes exponentially from its initial value to its final value (DP=1) or the level-off basic degree of polymerization with a linear and square time dependence.

Keywords: Cellulose, Degree of polymerization, Analytical solution, Level-off basic degree of polymerization

1. Introduction

Cellulose degradation is usually characterized in terms of degree of polymerization (DP) and its evolution is commonly described by the well known Ekenstam equation (Calvini 2008; Calvini 2012; Ding & Wang 2007; Ding & Wang 2008):

\[
\frac{d(DP)}{dt} = k(t)(DP - DP^2) \quad (1)
\]

\[
DP(0) = DP_0, k(0) = k_0 \quad (2)
\]

where \( DP_0 \) and \( DP \) are, respectively, the degree of polymerization before and after the degradation, \( k \) is the reaction rate depending on time. When \( k \) is a constant, \( k(t) = k_0 \), this equation admits an exact solution, which is (Calvini 2008)

\[
\ln(1 - \frac{1}{DP_0}) - \ln(1 - \frac{1}{DP}) = k_0 t \quad (3)
\]

The exact solution for other cases of \( k(t) \) is too complicated to give a direct insight into its meaning, and an approximate solution by analytical methods such as the variational iteration method (He 2012) and the homotopy perturbation method (He 2014) is, therefore, much needed.
2. Approximate solution

When the degradation evolution tends to infinity, we have the following equilibrium state

\[ k(t)(DP - DP^2) = 0, \quad t \to \infty \]  

(4)

That is

\[ DP(t \to \infty) = 1 \]  

(5)

Eq.(1) reveals that \( DP \) changes approximately exponentially from \( DP_0 \) at \( t=0 \) to a final value \( DP = 1 \) when time tends to infinity, accordingly we can assume \( DP \) can be expressed in the following form

\[ DP = 1 + (DP_0 - 1)\exp(-\sum_{n=1}^{N} a_n t^n) \]  

(6)

where \( a_n \) \( (n = 1 \sim N) \) are unknown constants to be further determined.

It is obvious that Eq.(6) satisfies the initial condition at \( t=0 \) and the terminal condition when \( t \to \infty \).

To show the solution process, we consider a simple case:

\[ DP = 1 + (DP_0 - 1)\exp(-a_1t - a_2t^2) \]  

(7)

From Eq.(1) we have

\[ DP'(0) = k_o(DP_0 - DP^2_0) \]  

(8)

and

\[ DP''(0) = k_o(DP''_0 - 2DP_0 DP'_0 + k'_o(DP_0 - DP^2_0) \\
= k^2_o(1 - 2DP_0)(DP_0 - DP^2_0) + k'_o(DP_0 - DP^2_0) \]  

(9)

Similarly from Eq.(7) we have

\[ DP'(t) = (DP_0 - 1)(-a_1 - 2a_2t)\exp(-a_1t - a_2t^2) \]  

(10)

\[ DP''(t) = (DP_0 - 1)(-2a_2 + (-a_1 - 2a_2t^2)\exp(-a_1t - a_2t^2) \]  

(11)

Eqs.(10) and (11) imply that

\[ DP'(0) = (1 - DP_0)a_1 \]  

(12)

\[ DP''(0) = (-a_1^2 + 2a_2)(1 - DP_0) \]  

(13)

Comparing Eqs.(12), (13) with Eqs.(8), (9) results in

\[ DP'(0) = k_o(DP_0 - DP^2_0) = (1 - DP_0)a_1 \]  

(14)
\[ DP'(0) = \dot{k}_0^2 (1 - 2DP_0)(DP_0 - DP_0^2) + k_0'(DP_0 - DP_0^2) = (-a_1^2 + 2a_2)(1 - DP_0) \quad (15) \]

Solving \( a_1 \) and \( a_2 \) from Eqs.(14) and (15), we have
\[ a_1 = k_0DP_0 \quad (16) \]
\[ a_2 = \frac{1}{2} \left[ k_0^2 (1 - DP_0) + k_0' \right] DP_0 \quad (17) \]

We obtain the following kinetic law
\[ DP = 1 + (DP_0 - 1) \exp \left[ -k_0DP_0 t - \frac{1}{2} \left[ k_0^2 (1 - DP_0) + k_0' \right] DP_0 t^2 \right] \quad (18) \]

Eq.(18) reveals that \( DP \) changes exponentially with a linear and square time dependence, from \( DP = DP_0 \) at initial value to \( DP = 1 \) at final version. By a similar operation, we can identify \( a_n \) \((n = 1 \sim N)\) in Eq.(6).

3. Discussions and conclusions

We propose a way to improve the kinetic description of cellulose degradation given by the Ekenstam equation, \( DP = 1 \) at infinite time is considered. This may be true, however, for pure hydrolysis of cellulose with sulfuric acid to the monomeric sugars, but normally cellulose degradation does not follow this path all the way to the monomer but stops at the level-off \( DP \) which is different from \( DP = 1 \). Eq.(6) is then can be updated as
\[ DP = \bar{DP} + (DP_0 - \bar{DP}) \exp \left( -\sum_{n=1}^{N} a_n t^n \right) \quad (19) \]

where \( \bar{DP} \) is the level-off basic degree of polymerization.

On the basis of the classical concept of cellulose structure, the true limiting degree of polymerization of cellulose is, of course, equal to 1. In other words, if the hydrolysis of pure cellulose was continued to completion, the end product of the hydrolysis would be mainly glucose. The \( \bar{DP} \), on the other hand, may vary over a range extending upward from a lower limit that corresponds to the maximum length of the hydrocellulose particle fragment that will dissolve in the particular hydrolyzing medium used (Cran, et al., 2011; Battista, et al., 1956; Fu, et al. 2012, Xing, et al., 2012). The kinetic law, Eq.(18), can be modified as
\[ DP = \bar{DP} + (DP_0 - \bar{DP}) \exp \left[ -k_0DP_0 t - \frac{1}{2} \left[ k_0^2 (1 - DP_0) + k_0' \right] DP_0 t^2 \right] \quad (20) \]

Eq.(20) illustrates that \( DP \) changes exponentially with a linear and square time dependence, from \( DP = DP_0 \) at initial value to \( DP = \bar{DP} \), the level-off basic degree of polymerization, at final version.

We obtained a kinetic law Eq.(18) for the hydrolysis of pure cellulose, and Eq.(20) for level-off basic degree of polymerization, revealing that an exponential change of \( DP \) from its initial value to its
final value, either for ideal case or $\bar{DP}$ for level-off DP. The mathematical analysis hereby can be easily extended to any complex processes, and the present paper can be used as a paradigm for many other applications in searching for analytical solutions for various cellulose degradations.

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