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3. New methods (analytical method, numerical method, optimization method, statistical method, allometric method, and others) for nonlinear equations. Generally one example should be given, sometimes only the solution procedure is enough.
4. New interpretation of a nonlinear phenomenon or a new solution of a nonlinear equation.
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Heat transfer analysis in diverging and converging channels

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Abstract

Heat transfer analysis of Jeffery-Hamel flow is presented; conservation laws along with similarity transforms has been used to determine the differential equations which govern the flow. Variational Iteration Method (VIM) is then employed to solve the re-formulated system of equations. For the sake of comparison a numerical solution is also presented. A concrete analysis of the parameters affecting the flow is given and a graphical demonstration is also provided. Jeffery Hamel Flows bear great importance due to their practical application in industrial and biological sciences.

Keywords: Jeffery-Hamel flows; Heat transfer; converging and diverging channels; Lagrange multiplier, nonlinear problems.

1. Introduction

After the founding investigations done by Jeffery and Hamel [1-2] over the flow between two non-parallel walls; this problem has got considerable attention from the scientists due to its wide range of applications that include aerospace, chemical, civil, environmental, mechanical, bio-mechanical engineering. Rivers and channels are also drastically simplified models described by this work.

Variation in angle between the walls and other flow parameters with their effects on fluid flow has been discussed extensively in [3-6]. To investigate these types of flows in a better way many studies have been carried out and we can find more flexible and explained material in literature now days, however many areas are still open and researcher are contributing to inspect the problem further; heat transfer analysis is one of them temperature distribution effects the flow behavior in many cases so to understand its effects on the flow heat transfer analysis is essential.

This article studies the heat transfer effects on Jeffery Hamel flows. Laws of conservation of momentum and energy lead us to a flow model describing the flow characteristics. Partial differential equations obtained are then converted in to nonlinear ordinary differential equations using similarity analysis. The Effects of different parameters on velocity and temperature profiles are discussed and their effects on both diverging and converging channels are presented. Due to nonlinearity of the equations, exact solutions are unlikely; so, a reliable and efficient technique called Variational Iteration Method (VIM) [7-10] is used to solve the re-formulated system of equations. VIM is a strong analytical

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technique and has been employed by several researchers in recent times to study different type of problems [11-16]. The originator of this highly reliable scheme is a Chinese Mathematician Ji- Huan He [8-17] who realized the real potential of this technique and converted it into an appropriate iterative scheme. The main positive features of this technique [8-28] is its simplicity, selection of initial approximation, compatibility with the nonlinearity of physical problems of diversified complex nature, minimal application of integral operator and rapid convergence [28].

2. Governing Equations

In this problem, we have considered an incompressible viscous fluid flow due to source or sink at the intersection of two rigid plane walls, angle between walls is $2\alpha$. Flow is assumed to be symmetric and purely radial. Under these assumptions velocity field take the form $V = [u_r, 0, 0]$, where $u_r$ is a function of both $r$ and $\theta$. The equation of continuity, motion and energy in polar coordinates in the absence of body forces under imposed assumptions become

$$\frac{1}{r} \frac{\partial}{\partial r} (ru_r) = 0,$$  \hspace{1cm} (1)

$$u_r \frac{\partial u_r}{\partial r} = -\frac{1}{\rho} \frac{\partial p}{\partial r} + \nu \left[ \frac{\partial^2 u_r}{\partial r^2} + \frac{1}{r} \frac{\partial u_r}{\partial r} + \frac{1}{r^2} \frac{\partial^2 u_r}{\partial \theta^2} - \frac{u_r}{r^2} \right],$$  \hspace{1cm} (2)

$$- \frac{1}{\rho r} \frac{\partial p}{\partial r} + 2\nu \frac{\partial u_r}{\partial \theta} = 0,$$  \hspace{1cm} (3)

$$\rho c_p u_r \frac{\partial T}{\partial r} = k \left[ \frac{\partial^2 T}{\partial r^2} + \frac{1}{r} \frac{\partial T}{\partial r} + \frac{1}{r^2} \frac{\partial^2 T}{\partial \theta^2} \right] + \mu \left[ 4 \left( \frac{\partial u_r}{\partial r} \right)^2 + \frac{1}{r^2} \left( \frac{\partial u_r}{\partial \theta} \right)^2 \right].$$  \hspace{1cm} (4)

Supporting boundary conditions are,\[ u_r = U, \quad \frac{\partial u_r}{\partial \theta} = 0, \quad \frac{\partial T}{\partial \theta} = 0 \quad \text{at} \quad \theta = 0, \]

$$u_r = 0, \quad T = T_w \quad \text{at} \quad \theta = \alpha$$  \hspace{1cm} (5)

where, $T_w$ is the temperature at the wall, $\gamma_1$ and $\delta_1$ is the velocity slip and thermal slip parameters respectively.

From the continuity equation (1), we can write

$$f(\theta) = ru_r(r, \theta).$$  \hspace{1cm} (6)

Using the dimensionless parameters

$$F(\eta) = \frac{f(\eta)}{f_{\text{max}}}, \quad \eta = \frac{\theta}{\alpha}, \quad \beta(\eta) = \frac{T}{T_w}$$  \hspace{1cm} (7)
Eliminating \( p \) from Eqs.(2) and (3) and using Eqs.(6) and (7), we get a system of nonlinear ordinary differential equation for the normalized velocity profile \( F(\eta) \) and temperature profile \( \beta(\eta) \),

\[
F^{''}(\eta) + 2\alpha \text{Re} F(\eta) F'(\eta) + 4\alpha^2 F'(\eta) = 0, \quad (8)
\]

\[
\beta^{''}(\eta) + Ec \text{Pr}[4\alpha^2 F^2(\eta) + (F'(\eta))^2] = 0. \quad (9)
\]

Using Eq. (6) and (7), the boundary conditions (5) will become

\[
F(0) = 1, \quad F'(0) = 0, \quad F(1) = 0
\]

\[
\beta(1) = 1, \quad \beta'(0) = 0, \quad (10)
\]

where Re is Reynolds number given by:

\[
\text{Re} = \frac{f}{\nu} = \frac{U r}{\nu} \alpha \begin{cases} \text{Divergent Channel : } \alpha > 0, U > 0 \\ \text{Convergent Channel : } \alpha < 0, U < 0 \end{cases}
\]

And

\[
Ec = \frac{\mu c_p}{k}, \quad \text{Pr} = \frac{U^2}{c_p T_w}
\]

Represent Eckert number and Prandtl number, respectively.

3. Solution Procedure

To solve Eqs.(9) and (10) with associated boundary conditions (11) using standard procedure for VIM, the correctional functional for the coupled system is given by

\[
F_{n+1}(\eta) = F_n(\eta) + \int_0^\eta \lambda_F(s) \left[ F^{''}(s) + 4\alpha^2 \tilde{F}'(s) + 2\alpha \text{Re} \tilde{F}(s) \tilde{F}'(s) \right] ds,
\]

\[
\beta_{n+1}(\eta) = \beta_n(\eta) + \int_0^\eta \lambda_\beta(s) \left[ \beta^{''}(s) + Ec \text{Pr}[4\alpha^2 \tilde{F}^2(\eta) + (\tilde{F}'(\eta))^2] \right] ds,
\]

where, \( \lambda_F(s) \) and \( \lambda_\beta(s) \) are Lagrange multipliers for velocity and temperature profile respectively. We can get approximate Lagrange multipliers as \( \lambda_F(s) = -\frac{(s-\eta)^2}{2!} \) and \( \lambda_\beta(s) = (s-\eta) \) so that the iterative formula (13) can be written as
\begin{equation}
F_{n+1}(\eta) = F_n(\eta) - \int_0^\eta \left( s - \eta \right)^2 \left( F^{'''}(s) + 4\alpha^2 F'(s) + 2\alpha \text{Re} F(s) F'(s) \right) ds,
\end{equation}
\begin{equation}
\beta_{n+1}(\eta) = \beta_n(\eta) + \int_0^\eta \left( s - \eta \right) \left( \frac{\rho'(\eta)}{\rho'(\eta)} + Ec Pr \left( 4\alpha^2 F^2(\eta) + (F'(\eta))^2 \right) \right) ds.
\end{equation}

Using boundary conditions given in Eq. (11), we can get
\begin{equation}
F_0(\eta) = 1 + A \frac{\eta^2}{2},
\end{equation}
\begin{equation}
\beta_0(\eta) = B,
\end{equation}
where A and B are constants to be determined by using boundary conditions \( F(1) = 0 \) and \( \beta(1) = 1 \), respectively.

Next few iterations of the solution are given by
\begin{equation}
F_1(\eta) = 1 + \frac{1}{2} A \eta^2 - \frac{1}{12} \eta^4 \alpha RA - \frac{1}{6} \eta^4 A - \frac{1}{120} \alpha RA^2 \eta^6,
\end{equation}
\begin{equation}
\beta_1(\eta) = B - 2 EP \alpha^2 \eta^2 + \left( -\frac{1}{3} EP \alpha^2 A - \frac{1}{12} EPA^2 \right) \eta^4
\end{equation}
\begin{equation}
+ \left( \frac{1}{45} EP \alpha RA^2 + \frac{1}{45} EP \alpha^3 RA + \frac{2}{45} EP \alpha^4 A + \frac{1}{90} EP \alpha^2 A^2 \right) \eta^6
\end{equation}
\begin{equation}
+ \left( -\frac{1}{1260} EP \alpha^3 RA^2 + \frac{1}{252} EP \alpha^4 A^2 + \frac{1}{560} EPA^3 \alpha R - \frac{1}{504} EP \alpha^2 R^2 A^2 \right) \eta^8
\end{equation}
\begin{equation}
+ \left( -\frac{1}{52800} EP \alpha^2 R^2 A^4 - \frac{1}{23760} EP \alpha^4 R^2 A^3 - \frac{1}{11880} EP \alpha^5 RA^3 \right) \eta^{10}
\end{equation}
\begin{equation}
+ \left( -\frac{1}{810} EP \alpha^5 RA^2 - \frac{1}{810} EP \alpha^6 A^2 - \frac{1}{2700} EP \alpha^2 R^2 A^3 \right) \eta^{10}
\end{equation}
\begin{equation}
- \frac{1}{2700} EP \alpha^3 R^3 A^3 - \frac{1}{3240} EP \alpha^4 R^2 A^2 \right) \eta^{10}
\end{equation}
\begin{equation}
- \frac{1}{655200} EP \alpha^4 R^2 A^4 \eta^{14}
\end{equation}
\[ F_2(\eta) = 1 + \frac{1}{2} A \eta^2 + \left( -\frac{1}{12} \alpha RA - \frac{1}{6} A \alpha^2 \right) \eta^4 + \left( \frac{1}{180} \alpha^2 R^2 A + \frac{1}{45} \alpha^3 RA - \frac{1}{120} \alpha RA^2 + \frac{1}{45} A \alpha^4 \right) \eta^6 + \left( \frac{1}{280} \alpha^3 RA^2 + \frac{1}{560} \alpha^2 R^2 A^2 \right) \eta^8 + \left( -\frac{1}{12960} \alpha^3 R^3 A^2 - \frac{1}{3240} \alpha^5 RA^2 + \frac{1}{10800} \alpha^2 R^2 A^3 - \frac{1}{3240} \alpha^4 R^2 A^2 \right) \eta^{10} + \left( -\frac{1}{95040} \alpha^3 R^3 A^3 - \frac{1}{47520} \alpha^4 R^2 A^3 \right) \eta^{12} - \frac{1}{2620800} \alpha^3 R^3 A^4 \eta^{14} \] (17)

\[ \beta_2(\eta) = B - 2 EP \alpha^2 \eta^2 + \left( -\frac{1}{3} EP \alpha^2 A - \frac{1}{12} EPA^2 \right) \eta^4 + \left( \frac{1}{45} EP \alpha RA^2 + \frac{1}{45} EP \alpha^3 RA + \frac{2}{45} EP \alpha^4 A + \frac{1}{90} EP \alpha^2 A^2 \right) \eta^6 + \left( -\frac{1}{1260} EP \alpha^4 A^2 - \frac{1}{315} EP \alpha^6 A - \frac{1}{1260} EP \alpha^4 R^2 A \right. \\
+ \left. \frac{1}{560} EPA^3 \alpha R - \frac{1}{180} EPA^3 RA^2 - \frac{1}{315} EP \alpha^5 RA - \frac{1}{315} EP \alpha^2 R^2 A^2 \right) \eta^8 + \cdots \] (18)

Similarly, other iterations for the solution can be obtained.

4. Results and discussions

Fig. 1 shows the physical behavior of the flow under varying angle \( \alpha \) in case of diverging channel, clearly velocity is observed to be decreasing function of angle \( \alpha \). Fig. 2 depicts the outcomes of increasing Reynolds number with a fixed value of angle \( \alpha \). It can be seen from Fig. 2 that the increment in Re results in decrement in velocity profile for a diverging channel. Maximum of the velocity is observed at the centerline of the channel for both cases.
Fig. 1: Variation of $F(\eta)$ for different values of $\alpha$ for diverging channel.

Fig. 2: Variation of $F(\eta)$ for different values of Re for diverging channel.

Fig. 3: Variation of $F(\eta)$ for different values of $\alpha$ for converging channel.

Fig. 4: Variation of $F(\eta)$ for different values of Re for converging channel.

Fig. 5: Variation of $\beta(\eta)$ for different values of $\alpha$ for diverging channel.

Fig. 6: Variation of $\beta(\eta)$ for different values of Re for diverging channel.

Fig. 3 and 4 illustrate the influence of angle $\alpha$ and Reynolds number on converging channel respectively. Behavior of flow for changing $\alpha$ and Re in converging channel is quite opposite to the behavior seen in diverging channel. But the maximum velocity at the centerline of channel appears as a common factor in both diverging and converging channels. Also, for $\alpha$ and Re velocity is a decreasing function.

Effects of angle $\alpha$, Reynolds number Re, Prandtl number Pr, Eckert number Ec and the on temperature profile for diverging channel are presented in Figs. 5-8, respectively. It can be observed clearly that
\( \beta(\eta) \) increases with increase in all \( \alpha \), \( \text{Re} \), \( \text{Pr} \), \( \text{Ec} \). The maximum value is observed to be at the centerline of the channel.

For a converging channel, variation in temperature behavior for varying flow parameters is shown in Figs. 9-12. Effects of \( \alpha \) and \( \text{Re} \) are opposite to each other in cases of narrowing and widening channel,
while Pr and Ec demonstrate similar effects in both cases. $\beta(\eta)$ decreases with an increase in $\alpha$ and Re while increases with an increase in Pr and Ec. Also, maximum value of $\beta(\eta)$ occurs at the centerline of the channel.

It is important to show that the series solutions given in Eq. (17) and Eq. (18) are convergent. Table.1 gives the convergence of these series solutions both for diverging and converging channels. Values of unknowns are obtained to check the convergence of the solution. From Table.1 it can be seen that only 6th order solution is enough to get a convergent solution.

### Table 1: Convergence of velocity and temperature profiles for $\alpha = 5^0$, $Re=80$, $Pr=1.0$ and $Ec=0.1$.

<table>
<thead>
<tr>
<th>Order of approximations</th>
<th>For Diverging Channel</th>
<th>For Converging Channel</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$A = -F^*(0)$</td>
<td>$B = \beta(0)$</td>
</tr>
<tr>
<td></td>
<td>$B = \beta(0)$</td>
<td>$A = -F^*(0)$</td>
</tr>
<tr>
<td>1</td>
<td>4.920640</td>
<td>1.048504</td>
</tr>
<tr>
<td>2</td>
<td>4.742404</td>
<td>1.052498</td>
</tr>
<tr>
<td>3</td>
<td>4.852013</td>
<td>1.053979</td>
</tr>
<tr>
<td>4</td>
<td>4.844710</td>
<td>1.053869</td>
</tr>
<tr>
<td>5</td>
<td>4.845071</td>
<td>1.053875</td>
</tr>
<tr>
<td>6</td>
<td>4.845071</td>
<td>1.053875</td>
</tr>
<tr>
<td>7</td>
<td>4.845071</td>
<td>1.053875</td>
</tr>
<tr>
<td>Numerical Solution</td>
<td>(4.845071)</td>
<td>(1.053875)</td>
</tr>
</tbody>
</table>

Same problem is solved by using a well-known numerical method, i.e. Runge-Kutta method (RK-4). Comparisons of VIM solution to numerical solution for velocity profile and temperature distribution in both diverging and converging channels are presented in Figs. 13-16. An excellent agreement between the solutions is found.

### Table 2: Comparison of numerical solution and VIM solution for diverging channel for $\alpha = 5^0$, $Re=80$, $Pr=1.0$ and $Ec=0.1$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>VIM</th>
<th>Numerical</th>
<th>VIM</th>
<th>Numerical</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1.053875</td>
<td>1.053875</td>
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<tr>
<td>0.1</td>
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<tr>
<td>0.2</td>
<td>0.907452</td>
<td>0.907452</td>
<td>1.053525</td>
<td>1.053525</td>
</tr>
<tr>
<td>0.3</td>
<td>0.803042</td>
<td>0.803042</td>
<td>1.052398</td>
<td>1.052398</td>
</tr>
<tr>
<td>0.4</td>
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<tr>
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<td>1.045519</td>
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<tr>
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<td>1</td>
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</table>

Table.2 gives a comparison of VIM solution and numerical solution for the case of diverging channel while in Table.3 same values are given for a converging channel. Both velocity and temperature profiles give an excellent agreement between two solutions. Figs. 13-16 depict a graphical representation for two solutions. Plane line gives the values for numerical solution while dotted line gives the same for VIM solution.
Table 3: Comparison of numerical solution and VIM solution for converging channel for $\alpha = 5^\circ$, $Re=80$, $Pr=1.0$ and $Ec=0.1$.

<table>
<thead>
<tr>
<th>$\eta$</th>
<th>VIM</th>
<th>Numerical</th>
<th>VIM</th>
<th>Numerical</th>
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Fig. 13: Comparison of numerical solution and VIM solution for $F(\eta)$ for diverging channel.

Fig. 14: Comparison of numerical solution and VIM solution for $\beta(\eta)$ for diverging channel.

Fig. 15: Comparison of numerical solution and VIM solution for $F(\eta)$ for converging channel.

Fig. 16: Comparison of numerical solution and VIM solution for $\beta(\eta)$ for converging channel.
5. Conclusions

This paper presents a study of Jeffrey Hamel flows with heat transfer. Resulting nonlinear equations are solved by Variational Iteration Method. Numerical solution is also obtained using Runge-Kutta order four (RK-4) method. Analytical and numerical solutions are compared. It is evident from tables and graphs that our results agree exceptionally well with the numerical results. Besides, from Figs. 1-16, we can conclude that:

I. For diverging channel ($\alpha > 0$) there is decrease in the velocity with the increase in angle $\alpha$ and Reynolds number Re.

II. The effect of angle $\alpha$ is quite opposite for converging channel ($\alpha < 0$) to that for diverging channel. There is an increase in the velocity for converging channel with an increase in $\alpha$ and Re.

III. Effect of Prandtl number and Eckert number is same for diverging and converging channel. There is an increase in temperature for both the cases. While $\alpha$ and Re have opposite effects for diverging and converging channel. Increase in temperature for the case of diverging channel is observed while for converging channel, temperature decreases.

IV. VIM and Numerical results are in exceptional agreement for both diverging and converging channel.

References


Collatz’ problem and the homotopy perturbation method

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Abstract

Collatz has used the problem of bending of a beam as a benchmark for various numerical schemes discussed in his classical work. In this communication we solve the same problem using the homotopy perturbation method which makes use of one auxiliary parameter. An optimum value of the parameter is determined and it is shown that with this value of the parameter, a sufficiently accurate solution is obtained using very few terms in the perturbation solution.

Keywords: Collatz’ problem, bending of a beam, homotopy perturbation method, auxiliary parameter, numerical solution

1. Introduction

The bending of a beam is a classical problem of elasticity, which does not admit a closed from analytical solution, and therefore has been used as a benchmark for various techniques, analytical and numerical, to compute approximate solutions. In particular, Collatz [1], in his celebrated work on numerical treatment of differential equations, extensively utilized the said problem to test the efficiency and accuracy of various schemes he pioneered. This was especially welcomed at a time when the digital computers had just started making an impact in the area of scientific computing. Collatz’ work embodied in the abovementioned monograph led to numerous algorithms which could be readily implemented on the computers and solutions obtained of problems which were hitherto unsolved because of lack of suitable powerful computing devices.

One of the algorithms advocated by Collatz consisted of introducing an auxiliary parameter in the trial solution that satisfies the boundary conditions of the problem. The problem was then solved afresh with the trial solution substituted in the differential equation. The auxiliary parameter was subsequently determined by requiring that the “size” of the newly obtained solution matched with that of the trial solution. It was found that the algorithm introduced by Collatz not only gave an analytical solution in the closed form but also that the solution so obtained was sufficiently accurate for engineering purposes. In particular, Collatz demonstrated the usefulness of the algorithm by applying it to the problem of bending of a beam. He further showed the flexibility inherent in the algorithm by considering its numerous variants and pointing out the accuracy obtained in each variant.

Ariel [2] utilized Collatz algorithm for computing an approximate solution of the pressure gradient driven flow of a third grade fluid in a porous channel. He showed that the approximate solution agreed well with the exact numerical solution. Further the approximate solution could predict the behavior of the solution for those parameters for which a numerical solution was not feasible. The velocity field derived by Ariel [2], using the Collatz’ method, was later utilized by Ellahi et al [3] to derive analytically the temperature distribution in the channel for the flow of a third grade fluid. Once again the analytical solution tallied very well with the numerical solution.
In the present communication we incorporate Collatz’ idea to obtain the solution of the problem of the bending of a beam by applying it in the context of the homotopy perturbation method (HPM), introduced and developed by He [4]. The HPM is one of the modern analytical methods which is highly versatile and allows accurate solutions of highly nonlinear physical problems in very few steps. It has been extensively invoked successfully recently in various areas of scientific and technological importance (see for example [5-14]). As we show presently an intelligent use of the HPM can drastically reduce the computational overhead and yield compact analytical expressions for the results that can be readily evaluated using the hand-held calculators.

2. Bending of a Beam

Consider a beam of length $2l$ placed horizontally and clamped at the two ends. A weight $W$ is attached to the center of the beam which causes the beam to be deflected vertically. If the center of the beam in the undisturbed position is taken as the origin, its original orientation along the axis of $x$, and the vertical direction through the origin along the axis of $y$-axis, then it can be shown that $y$, the deflection at a distance $x$ satisfies the following boundary value problem (BVP) in the non-dimensional form

$$\begin{align*}
y'' + (1 + x^2)y &= -1, \\
y'(0) &= 0, \quad y(1) = 0.
\end{align*}$$

(1)

(2)

Here a prime denotes the derivative with respect to $x$. In the derivation of equation (1), while calculating the curvature at a distance $x$, the squares of the slopes of the beam have been neglected, implying that equation (1) is valid only for small deflections.

Rather surprisingly BVP (1)-(2) does not admit a closed form solution in terms of elementary functions. Collatz, amongst several schemes, suggested an approximate solution of the BVP (1)-(2) in the form

$$y = A(1 - x^2),$$

(3)

which satisfies the boundary conditions of the problem.

Progressively aiming at more accurate solutions, Collatz gave three iterative schemes:

(i) $y''_{n+1} = -1 - (1 + x^2)y'_n,$

(4)

(ii) $y''_{n+1} + y_{n+1} = -1 - x^2 y_n,$

(5)

(iii) $y''_{n+1} + \frac{5}{4} y_{n+1} = -1 + \left(\frac{1}{4} - x^2\right)y_n,$

(6)

the rationale behind the last scheme being that $\frac{5}{4}$ is an average of $1 + x^2$.

By matching the values of $y$ at $x = 0$ at the zeroth and first iterations, Collatz was able to find the value of $A$. He further demonstrated that a careful shifting of the $y_n$ term can go a long way in improving the accuracy of the solution.

Motivated by Collatz’ suggestion and the success of the HPM method in obtaining the analytical solutions of various problems, we thought it prudent to combine the two ideas. As a result, in the following we introduce the Collatz variant of the HPM.
3. Collatz Variant of the HPM

Rather than transferring $y$ or $\frac{1}{2} y$ to the left hand side of equation (4), we transfer an arbitrary multiple of $y$. Thus we rewrite equation (1) as

$$y'' + \alpha^2 y + 1 = -(1 + x^2 - \alpha^2) y,$$

where $\alpha$ is a suitable auxiliary parameter whose value is to be determined in an appropriate manner in the course of the solution.

Setting up the HPM formulation for equation (7), we write

$$y'' + \alpha^2 y + 1 = -p(1 + x^2 - \alpha^2) y,$$

where $p$ is the homotopy parameter.

We seek a solution for $y$ in the form

$$y = y_0 + y_1 p + y_2 p^2 + \cdots = \sum_{n=0}^{\infty} y_n p^n.$$

Substituting for $y$ from equation (9) into equation (8), and equating like powers of $p$ on both sides we obtain

$$y_0'' + \alpha^2 y_0 = -1,$$

and

$$y_{n+1}'' + \alpha^2 y_{n+1} = -(1 + x^2 - \alpha^2) y_n, \quad n \geq 0.$$

The boundary conditions on $y$ are

$$y'_n(0) = 0, \quad y'_n(1) = 0, \quad n \geq 0.$$

The solution of the zeroth order solution is

$$y_0(x) = \frac{1}{\alpha^2} \left( 1 - \frac{\cos \alpha x}{\cos \alpha} \right).$$

The higher order solutions can be readily computed from BVP (11)-(12), though they become increasingly involved. For example, the first order solution is given by

$$y_1(x) = \frac{1}{12 \alpha^6 \cos^2 \alpha} \left[ \left( 24 \cos \alpha - 9 \alpha^2 + 12 \alpha^4 - 3 \alpha^2 x^2 \right) \cos \alpha - (3 - 2 \alpha^2 + 6 \alpha^4) \sin \alpha \right] \cos \alpha x + \left( 3 + 6 \alpha^4 - 2 \alpha^2 x^2 \right) \alpha x \cos \alpha \sin \alpha x - 12 \left( 2 + \alpha^4 - \alpha^2 x^2 \right) \cos^2 \alpha.$$

The solution for $y(x)$ is simply
Note that we still have the auxiliary parameter $\alpha$ in the solution. We need to estimate its value to get the final solution. There can be a number of techniques to estimate the value of $\alpha$. We decided to settle down for that value of $\alpha$ which, relatively speaking, stabilizes the value of $y(0)$. Accordingly in Figure 1, limiting ourselves to the third order terms in expansion (15), we have plotted $y(0)$ against $\alpha$. It can be seen from the figure that $y(0)$ remains almost stationary for $\alpha = 1.065$. Thus we note that the optimum value of $\alpha$ lies somewhere between the two choices made by Collatz, namely, 1 and 1.25.

![Figure 1. Illustrating the variation of $y(0)$ with $\alpha$, for the third order approximation](image)

With the value of $\alpha$ determined “optimally”, we can now list the successive approximations. We get

\[ y_0(x) = 1.819714\cos(1.065x) - 0.881659, \]  
\[ y_1(x) = (1.530077 - 0.401092x^2)\cos(1.065x) + (1.345610x - -0.284775x^3)\sin(1.065x) - 2.252328 + 0.777323x^2 \]

\[ y_2(x) = (20.455169 - 1.937358x^2 + 0.431597x^4 - 0.022283x^6)\cos(1.065x) + + (2.633882x - 1.121154x^3 + 0.100430x^5)\sin(1.065x) - - (19.909993 - 10.013883x^2 + 0.685334x^4), \]
\[ y_3(x) = (543.755754 - 14.105631x^2 + 3.180139x^4 - 0.330713x^6 + 
+ 0.010805x^8) \cos(1.065x) + (24.137101x - 9.517096x^3 + 1.159463x^5 - 
- 0.071225x^7 + 0.001162x^9) \sin(1.065x) - (544.166790 - 
- 297.312583x^2 + 25.495948x^4 - 0.604231x^6). \] (19)

Accordingly \( y(x) \) can be approximated by

\[ y(x) = (567.560713 - 16.444081x^2 + 3.611736x^4 - 0.352996x^6 + 
+ 0.010805x^8) \cos(1.065x) + (28.116593x - 10.923024x^3 + 1.259893x^5 - 
- 0.071225x^7 + 0.001162x^9) \sin(1.065x) - (567.210770 - 
- 308.103789x^2 + 26.181282x^4 - 0.604231x^6). \] (20)

**Table 1** Illustrating the variation of \( y(x) \) with \( x \), for the first three approximations. The exact solution is given in the last column.

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In Table 1, the values of \( y(x) \) are presented against \( x \) at the first three approximations. Also given in the last column is the exact value of \( y(x) \) obtained numerically by Numerov’s method. It can be seen that there is an excellent agreement between the three-term solution obtained by the HPM method and the exact numerical solution.

4. Conclusion

In the present work, we have incorporated the Collatz’ idea of having an auxiliary parameter in the trial solution within the framework of the HPM for deriving an approximate solution of a boundary value problem. The problem of the bending of a beam under the influence of a weight attached at the center of the beam is considered. By incorporating an auxiliary parameter along the lines of Collatz’ suggestion for obtaining an approximate solution iteratively, with the HPM, and estimating the value of auxiliary parameter by looking at the stationary value of the deflection in the center, an accurate analytical solution has been derived, which is in excellent agreement with the exact numerical solution.

References

Analytic solutions for a nonlinear Schrödinger-type system

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Abstract

An analytic study on a nonlinear Schrödinger-type system called the generalized Zakharov system is presented in this paper. The (G'/G)-expansion method and the Exp-function method are employed to construct exact travelling wave solutions of this system in forms of the hyperbolic functions and the trigonometric functions. Some of solutions gained from each of the proposed methods have been compared and verified together.

MSC (2010) No.: 35Q55; 35C07; 35Q60; 37N20

Keywords: Nonlinear Schrödinger equations; Generalized Zakharov system; (G'/G)-expansion method; Exp-function method; Soliton; Periodic solutions

1. Introduction

In the first of his four lectures on wave mechanics, Schrödinger wrote:

“Substituting from (12) and (8) in (10) and replacing p by ψ (…) we obtain

\[ \nabla^2 \psi + \frac{8\pi^2 m}{\hbar^2} (E - V) \psi = 0. \]

A simplification in the problem of the “mechanical waves” consists in the absence of boundary conditions. I thought the latter simplification fatal when I first attacked these equations. Being insufficiently versed in mathematics, I could not imagine how proper vibration frequencies could appear without boundary conditions. Later on I recognized that the more complicated form of the coefficients (i.e. the appearance of V(x,y,z)) takes charge, so to speak, of what is ordinarily brought about by boundary conditions, namely, the selection of definite values of E.” (Dr. Erwin Schrödinger – Four Lectures on Wave Mechanics. Delivered at the Royal Institution, London, on 5, 7, and 14 March, 1928)

Here we are: the above equation arrived in ‘Knowledge Space’ and is there to stay. Who could have thought, though, that some 85 years later, people are still thinking about solving it faster and more accurately?

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As you know, up to now, various methods have been utilized to explore different kinds of solutions of physical models described by nonlinear partial differential equations (NPDEs). In the numerical methods [1], stability and convergence should be considered, so as to avoid divergent or inappropriate results. However, in recent years, a variety of effective analytical and semi-analytical methods have been developed to be used for solving nonlinear PDEs, such as the variational iteration method (VIM) [2,3], the homotopy perturbation method (HPM) [4,5], E-infinity theory [6], the parameter-expansion method [7], the sine-cosine method [8], the tanh method [8], the homotopy analysis method (HAM) [9,10], the homogeneous balance method [11], the inverse scattering method [12], and others. Likewise, He and Wu [13] proposed a straightforward and concise method called the Exp-function method to obtain solitary wave solutions, periodic solutions and compact-like solutions of NEEs. The basic idea of the Exp-function method was presented in He’s monograph [14]. The method, with the aid of Maple or Matlab, has been successfully applied to many kinds of NEEs [15-23]. Lately, the (G'/G)-expansion method, first introduced by Wang et al. [24], has become widely used to search for various exact solutions of NEEs [17, 18, 25-30]. The results reveal that the two recent methods are powerful techniques for solving NPDEs in terms of accuracy and efficiency. This is important, since systems of NPDEs have many applications in engineering.

In the interaction of laser-plasma, the system of Zakharov equations plays an important role. The Zakharov system is a type of nonlinear Schrödinger equations. This system attracted many scientists’ wide interest and attention. The generalized Zakharov system can be given as

\[
i \psi_t + \psi_{xx} - 2\gamma |\psi|^2 \psi + 2\mu \psi = 0,
\]

\[
\frac{1}{c^2} \nu_t - \nu_{xx} + \mu (|\psi|^2)_{xx} = 0.
\]

where the real unknown function \( \nu(x,t) \) is the fluctuation in the ion density about its equilibrium value, and the complex unknown function \( \psi(x,t) \) is the slowly varying envelope of highly oscillatory electron field. The parameters \( \gamma, \mu \) and \( c \) are real numbers, where \( c \) is proportional to the electron sound speed. The coefficient \( \gamma \) is a real constant that can be a positive or negative number. The general form of (1.1) and (1.2) covers many generalized Zakharov systems arising in various physical applications. The well-known Zakharov system (ZS) has been first derived by Zakharov [31] to describe the interaction between Langmuir (dispersive) and ion acoustic (approximately non-dispersive) waves in a plasma. Later, it has become commonly accepted that the ZS is a general model to govern interaction of dispersive and non-dispersive waves. When \( \gamma = 0 \) and \( \mu = 1 \), this system is reduced to the classical Zakharov system of plasma physics. When the sound speed \( c \to +\infty \), the so-called subsonic limit, the Zakharov system becomes the cubically nonlinear Schrödinger equation. If we set \( c = 1 \) and \( \mu = 1 \), the generalized Zakharov system becomes [8,32]

\[
i \psi_t + \psi_{xx} - 2\gamma |\psi|^2 \psi + 2\psi = 0,
\]

\[
\nu_t - \nu_{xx} + (|\psi|^2)_{xx} = 0.
\]

Up to now, many numerical and analytical methods have been proposed to search some types of solutions of the ZS. For example, Payne et al. [33] designed a spectral method for a 1D ZS. They used a truncated Fourier expansion in their scheme to eliminate the aliasing errors. Glassey [34] presented an energy-preserving finite difference scheme for the ZS in one dimension, and proved its
convergence in [35]. Also, Chang et al. [36] applied a conservative difference scheme for the generalized Zakharov system. This scheme can be implicit or semi-explicit depending on the choice of a parameter. They also proved the convergence of their method. In two other numerical studies, Bao et al. [37] and Jin et al. [32] proposed two time-splitting spectral techniques to solve the generalized ZS. In recent years, several analytical methods have been used to find exact travelling wave solutions of the generalized ZS. El-Wakil et al. [8] applied the extended tanh and the sine-cosine methods to look for exact periodic and soliton solutions of the system. Lately, Taghizadeh et al. [38] implemented the infinite series method for obtaining exact solutions of Eqs. (1.3) and (1.4). In another study, Al-Muhiameed and Abdel-Salam [39] used an improved Jacobi elliptic function method to derive exact solutions of the ZS with the aid of the homogenous balance principle.

Considering all the indispensably significant issues mentioned above, the objective of this paper is to investigate the travelling wave solutions of the generalized Zakharov system, Eqs. (1.3) and (1.4) systematically, by applying the Exp-function and the (G'/G)-expansion methods. Some of solutions obtained by each of the mentioned methods have been verified together.

2. Description of the two methods

2.1. The (G'/G)-expansion method

Suppose that a nonlinear PDE, say in two independent variables x and t, is given by

\[ P(u, u_x, u_{xx}, u_{xxx}, \ldots) = 0, \]  

(2.1)

where P is a polynomial in its arguments, which include nonlinear terms and the highest-order derivatives.

Introducing a wave variation \( \eta \) defined as

\[ u(x, t) = U(\eta), \quad \eta = k(x - ct) \]  

(2.2)

Eq. (2.1) reduces to the ordinary differential equations (ODE)

\[ P(U, -kcU', kU', k^2U'', k^2c^2U''', -k^2cU''', \ldots) = 0, \]  

(2.3)

in which \( k \) and \( c \) are constants to be determined later. According to the (G'/G)-expansion method, it is assumed that the travelling wave solution of Eq. (2.3) can be expressed by a polynomial in \( \frac{G'}{G} \) as follows:

\[ U(\eta) = \sum_{i=1}^{m} \alpha_i \left( \frac{G'}{G} \right)^i + \alpha_0, \quad \alpha_m \neq 0 \]  

(2.4)

where \( \alpha_0 \) and \( \alpha_i \), for \( i = 1, 2, \ldots, m \), are constants to be determined later, \( G(\eta) \) satisfies a second-order linear ordinary differential equation (LODE):

\[ \frac{d^2 G(\eta)}{d\eta^2} + \lambda \frac{dG(\eta)}{d\eta} + \mu G(\eta) = 0 \]  

(2.5)
where $\lambda$ and $\mu$ are arbitrary constants. Using the general solutions of Eq. (2.5), we have

\[
\frac{G'(\eta)}{G(\eta)} = \begin{cases}
\sqrt{\lambda^2 - 4\mu} \frac{C_1 \sinh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta\right) + C_2 \cosh\left(\frac{\sqrt{\lambda^2 - 4\mu}}{2} \eta\right)}{2} - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\
\sqrt{4\mu - \lambda^2} \frac{C_1 \sin\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \eta\right) + C_2 \cos\left(\frac{\sqrt{4\mu - \lambda^2}}{2} \eta\right)}{2} - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0,
\end{cases}
\] (2.6)

and it follows from (2.4) and (2.5) that

\[
U' = -\sum_{i=1}^{m} i\alpha_i \left[ \left(\frac{G'}{G}\right)^{i+1} + \lambda \left(\frac{G'}{G}\right)^i + \mu \left(\frac{G'}{G}\right)^{i-1} \right],
\]

\[
U'' = \sum_{i=1}^{m} i\alpha_i \left[ (i+1) \left(\frac{G'}{G}\right)^{i+2} + \lambda (2i+1) \left(\frac{G'}{G}\right)^{i+1} + i (\lambda^2 + 2\mu) \left(\frac{G'}{G}\right)^i \right]
\] (2.7)

and so on. Here, the prime denotes the derivative with respective to $\eta$.

To determine $u$ explicitly, we take the following four steps:

**Step 1.** Determine the integer $m$ by substituting Eq. (2.4) along with Eq. (2.5) into Eq. (2.3), and balancing the highest-order nonlinear term(s) and the highest-order partial derivative.

**Step 2.** Substitute Eq. (2.4) with the value of $m$ determined in Step 1, along with Eq. (2.5) into Eq. (2.3) and collect all terms with the same order of $\left(\frac{G'}{G}\right)$ together; the left-hand side of Eq. (2.3) is converted into a polynomial in $\left(\frac{G'}{G}\right)$. Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for $k, c, \alpha_0$ and $\alpha_i$, for $i = 1, 2, \ldots, m$.

**Step 3.** Solve the system of algebraic equations obtained in Step 2, for $k, c, \alpha_0$ and $\alpha_i$, for $i = 1, 2, \ldots, m$, by use of Maple.

**Step 4.** Use the results obtained in the above steps to derive a series of fundamental solutions $u(\eta)$ of Eq. (2.3) depending on $\left(\frac{G'}{G}\right)$; since the solutions of Eq. (2.5) are well known to us, we can obtain exact solutions of Eq. (2.1).
2.2. The Exp-function method

According to the classic Exp-function method, it is assumed that the solution of ODE (2.3) can be written as

\[ U(\eta) = \sum_{n=-c}^{d} a_n \exp(n\eta) + \sum_{m=-f}^{g} b_m \exp(m\eta) = a_c \exp(c\eta) + ... + a_{-d} \exp(-d\eta) + b_f \exp(f\eta) + ... + b_{-g} \exp(-g\eta). \]  

(2.8)

where \( c, d, f \) and \( g \) are positive integers which are unknown, to be further determined, and \( a_n \) and \( b_m \) are unknown constants.

3. The Generalized Zakharov System

3.1. Application of the (G'/G)-expansion method

Let us assume the travelling wave solution of Eqs. (1.3) and (1.4) in the form

\[ \psi(x,t) = e^{\theta} u(\eta), \quad \nu(x,t) = \nu(\eta), \quad \theta = px + qt, \quad \eta = k(x - 2pt) \]  

(3.1)

where \( u(\eta) \) and \( \nu(\eta) \) are real functions, the constants \( p,q \) and \( k \) are to be determined later. Substituting (3.1) into Eqs. (1.3) and (1.4), we have

\[ k^2 u'' + 2u \nu - (p^2 + q) u - 2\gamma u^3 = 0, \]  

(3.2)

\[ k^2 (4p^2 -1) \nu' + k^2 (u^2)' = 0. \]  

(3.3)

In order to simplify ODEs (3.2) and (3.3), integrating Eq. (3.3) once and taking integration constant to zero, and integrating yields

\[ \nu(\eta) = \frac{-u^2}{1-4p^2} + c', \quad p^2 \neq \frac{1}{4} \]  

(3.4)

where \( c' \) -integration constant. Inserting Eq. (3.4) into (3.2), we obtain

\[ k^2 u'' + \left( 2c' - p^2 - q \right) u + 2\left( \frac{1}{1-4p^2} - \gamma \right) u^3 = 0. \]  

(3.5)

According to step 1, considering the homogeneous balance between \( u'' \) and \( u^3 \) in Eq. (3.5) gives
\[ m + 2 = 3m, \]  
so that
\[ m = 1. \]  

Suppose that the solutions of (3.5) can be expressed by a polynomial in \( \frac{G'}{G} \) as follows:
\[ u(\eta) = \alpha_0 + \alpha_1 \left( \frac{G'}{G} \right), \quad \alpha_1 \neq 0, \]  
where \( \alpha_0 \) and \( \alpha_1 \) are constants which are unknown, to be determined later.

Substituting Eq. (3.8) along with Eq. (2.5) into Eq. (3.5) and collecting all terms with the same power of \( \frac{G'}{G} \) together, the left-hand side of Eq. (3.5) is converted into a polynomial in \( \frac{G'}{G} \).

Equating each coefficient of this polynomial to zero yields a set of simultaneous algebraic equations for \( \alpha_0, \alpha_1, k, p, q, c', \lambda \) and \( \mu \). Solving the system of algebraic equations with the aid of Maple 13, we obtain the following:
\[ k = \pm \frac{2\alpha_0}{\lambda} \sqrt{\gamma + \frac{1}{4p^2 - 1}}, \quad \alpha_0 = \alpha_0, \quad \alpha_1 = \frac{2\alpha_0}{\lambda}, \quad p = p, \quad c' = c', \]  
\[ q = \frac{\Psi}{\lambda^2(4p^2 - 1)} \]  
where \( p, c', \alpha_0, \lambda \) and \( \mu \) are arbitrary constants, and \( \Psi \) is given as
\[ \Psi = -4\lambda^2 p^4 - 8\gamma \alpha_0^2 \lambda^2 p^2 + 8c' \lambda^2 p^2 + \lambda^2 p^2 + 32\mu \gamma \alpha_0^2 p^2 - 2\lambda^2 \alpha_0^2 + 2\gamma \lambda^2 \alpha_0^2 + 8\mu \gamma \alpha_0^2 - 2c' \lambda^2 - 8\mu \gamma \alpha_0^2. \]

By using Eq. (3.9), expression (3.7) can be written as
\[ u(\eta) = \alpha_0 \left[ 1 + \frac{2}{\lambda} \left( \frac{G'}{G} \right) \right], \]  
in which \( \eta = k(x - 2pt) = \pm \frac{2\alpha_0}{\lambda} \sqrt{\gamma + \frac{1}{4p^2 - 1}} (x - 2pt) \) and \( \gamma + \frac{1}{4p^2 - 1} > 0 \).

Inserting the general solution of (2.6) into Eq. (3.10), we get the generalized travelling wave solutions as follows:

Case A. \( \lambda^2 - 4\mu > 0 \).
\[ u(x,t) = \frac{\alpha_0 \sqrt{\lambda^2 - 4 \mu}}{\lambda} \left( \frac{C_1 \sinh(\xi) + C_2 \cosh(\xi)}{C_1 \cosh(\xi) + C_2 \sinh(\xi)} \right), \quad (3.11) \]

where \( \xi = \frac{\sqrt{\lambda^2 - 4 \mu}}{2} \eta = \pm \frac{\alpha_0 \sqrt{\lambda^2 - 4 \mu}}{\lambda} \left( \sqrt{\gamma + \frac{1}{4p^2 - 1}} (x - 2pt) \right) \) and \( \gamma + \frac{1}{4p^2 - 1} > 0 \).

Substituting Eq. (3.11) into the transformations (3.1) and (3.4) leads to the following hyperbolic function solutions of Eqs. (1.3) and (1.4):

\[ \psi(x,t) = \frac{\alpha_0 \sqrt{\lambda^2 - 4 \mu}}{\lambda} \left( \frac{C_1 \sinh(\xi) + C_2 \cosh(\xi)}{C_1 \cosh(\xi) + C_2 \sinh(\xi)} \right) e^{i(px+qt)}, \quad (3.12) \]

and

\[ \psi(x,t) = \frac{u^2(x,t)}{1 - 4p^2} + c', \quad (3.13) \]

in which \( \lambda, \mu, c', \alpha_0 \) and \( p \) are arbitrary constants; and \( q \) should be recalled from Eq. (3.9).

Now, to obtain some special cases of the above general solutions, we set \( C_2 = 0 \); then the solutions (3.12) and (3.13) reduce to

\[ \psi(x,t) = \frac{\alpha_0 \sqrt{\lambda^2 - 4 \mu}}{\lambda} \tanh(\xi) e^{i(px+qt)} = \frac{\alpha_0 \sqrt{\lambda^2 - 4 \mu}}{\lambda} \tanh \left[ \pm \frac{\alpha_0 \sqrt{\lambda^2 - 4 \mu}}{\lambda} \left( \sqrt{\gamma + \frac{1}{4p^2 - 1}} (x - 2pt) \right) \right] e^{i(px+qt)}, \quad (3.14) \]

and

\[ \psi(x,t) = \frac{\alpha_0^2 \left( \lambda^2 - 4 \mu \right)}{\lambda^2 \left( 1 - 4p^2 \right)} \tanh^2 (\xi) + c' = \frac{\alpha_0^2 \left( \lambda^2 - 4 \mu \right)}{\lambda^2 \left( 1 - 4p^2 \right)} \tanh^2 \left[ \pm \frac{\alpha_0 \sqrt{\lambda^2 - 4 \mu}}{\lambda} \left( \sqrt{\gamma + \frac{1}{4p^2 - 1}} (x - 2pt) \right) \right] + c', \quad (3.15) \]

and, when \( C_1 = 0 \), our general solutions, (3.12) and (3.13), lead to
\[
\psi(x,t) = \frac{\alpha_0 \sqrt{\lambda^2 - 4\mu}}{\lambda} \coth(\xi) e^{i(px + qt)} = \frac{\alpha_0 \sqrt{\lambda^2 - 4\mu}}{\lambda} \coth\left[ \pm \frac{\alpha_0 \sqrt{\lambda^2 - 4\mu}}{\lambda} \left( \sqrt{\frac{\gamma}{4p^2 - 1}} (x - 2pt) \right) \right] e^{i(px + qt)},
\]

(3.16)

and

\[
v(x,t) = \frac{\alpha_0^2 (\lambda^2 - 4\mu)}{\lambda^2 (1 - 4p^2)} \coth^2(\xi) + c' = \frac{\alpha_0^2 (\lambda^2 - 4\mu)}{\lambda^2 (1 - 4p^2)} \coth^2 \left[ \pm \frac{\alpha_0 \sqrt{\lambda^2 - 4\mu}}{\lambda} \left( \sqrt{\frac{\gamma}{4p^2 - 1}} (x - 2pt) \right) \right] + c'.
\]

(3.17)

Case B. \(\lambda^2 - 4\mu < 0\).

\[
u(x,t) = \frac{\alpha_0 \sqrt{2\lambda^2}}{\sqrt{4\mu - \lambda^2}} \left( \frac{-C_1 \sin(\xi) + C_2 \cos(\xi)}{C_1 \cos(\xi) + C_2 \sin(\xi)} \right),
\]

(3.18)
in which \(\xi = \sqrt{\frac{4\mu - \lambda^2}{2}} \eta = \pm \frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \left( \sqrt{\frac{\gamma}{4p^2 - 1}} (x - 2pt) \right)\) and \(\gamma + \frac{1}{4p^2 - 1} > 0\).

Inserting Eq. (3.18) into the transformations (3.1) and (3.4), we obtain the generalized trigonometric function solutions of Eqs. (1.3) and (1.4) as follows:

\[
\psi(x,t) = \frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \left( \frac{-C_1 \sin(\xi) + C_2 \cos(\xi)}{C_1 \cos(\xi) + C_2 \sin(\xi)} \right) e^{i(px+qt)},
\]

(3.19)

and

\[
u(x,t) = \frac{\alpha_0^2 (\lambda^2 - 4\mu)}{\lambda^2 (1 - 4p^2)} \coth^2 \left( \pm \frac{\alpha_0 \sqrt{\lambda^2 - 4\mu}}{\lambda} \left( \sqrt{\frac{\gamma}{4p^2 - 1}} (x - 2pt) \right) \right) + c'.
\]

(3.20)
in which \(\lambda, \mu, c', \alpha_0\) and \(p\) are arbitrary constants; and \(q\) should be recalled from Eq. (3.9).

Similarly, to derive some special cases of the above general solutions, we choose \(C_2 = 0\); then the solutions (3.19) and (3.20) lead to
\[ \psi(x,t) = -\frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \tan(\xi) e^{i(px + qt)} = \]
\[ -\frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \tan \left( \pm \frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \left( \sqrt{\gamma + \frac{1}{4p^2 - 1}} (x - 2pt) \right) \right) e^{i(px + qt)}, \]

and

\[ v(x,t) = \frac{\alpha_0^2 (4\mu - \lambda^2)}{\lambda^2 (1 - 4p^2)} \tan^2(\xi) + c' = \]
\[ \frac{\alpha_0^2 (4\mu - \lambda^2)}{\lambda^2 (1 - 4p^2)} \tan^2 \left( \pm \frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \left( \sqrt{\gamma + \frac{1}{4p^2 - 1}} (x - 2pt) \right) \right) + c', \]

and, when \( C_1 = 0 \), our general solutions, (3.19) and (3.20), reduce to

\[ \psi(x,t) = \frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \cot(\xi) e^{i(px + qt)} = \]
\[ \frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \cot \left( \pm \frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \left( \sqrt{\gamma + \frac{1}{4p^2 - 1}} (x - 2pt) \right) \right) e^{i(px + qt)}, \]

and

\[ v(x,t) = \frac{\alpha_0^2 (4\mu - \lambda^2)}{\lambda^2 (1 - 4p^2)} \cot^2(\xi) + c' = \]
\[ \frac{\alpha_0^2 (4\mu - \lambda^2)}{\lambda^2 (1 - 4p^2)} \cot^2 \left( \pm \frac{\alpha_0 \sqrt{4\mu - \lambda^2}}{\lambda} \left( \sqrt{\gamma + \frac{1}{4p^2 - 1}} (x - 2pt) \right) \right) + c'. \]

3.2. Application of the Exp-function method

In order to determine values of \( c \) and \( f \) in Eq. (2.8), we balance the linear term of the highest order \( u'' \) with the highest order nonlinear term \( u^3 \) in Eq. (3.5); we have

\[ u'' = \frac{c_1 \exp[(3f + c)\eta]}{c_2 \exp[4f\eta]} + \ldots, \]

(3.25)

\[ u^3 = \frac{c_3 \exp[(3c + f)\eta]}{c_4 \exp[4f\eta]} + \ldots, \]

(3.26)
where \( c_i \) are determined coefficients only for simplicity. Balancing the highest order of the Exp-function in Eqs. (3.25) and (3.26), we have

\[
3f + c = 3c + f,
\]

which leads to the result

\[
f = c.
\]

Similarly, to determine values of \( d \) and \( g \), we balance the linear term of the lowest order in Eq. (3.5)

\[
\eta^2 f + \eta \eta g = \eta^2 d + \eta g + d_1 \exp[-(3g + d)\eta] + d_2 \exp[-4g\eta],
\]

and

\[
\eta^3 f = \eta^3 d + \eta^3 g + d_3 \exp[-(3d + g)\eta] + d_4 \exp[-4g\eta],
\]

where \( d_i \) are determined coefficients for simplicity. Balancing the lowest order of the Exp-Function in Eqs. (3.29) and (3.30), we have

\[
-(3g + d) = -(3d + g),
\]

which leads to the result

\[
g = d.
\]

We can freely choose the values of \( p \) and \( q \). For simplicity, we set \( f = c = 1 \) and \( d = g = 1 \), so Eq. (2.8) reduces to

\[
\eta^2 f - \eta \eta g + \eta^2 d + \eta g + d_1 \exp[-(3g + d)\eta] + d_2 \exp[-4g\eta] = 0.
\]

Substituting Eq. (3.33) into Eq. (3.5), and making use of Maple, we arrive at

\[
\frac{1}{A} \left[ C_3 \exp(3\eta) + C_2 \exp(2\eta) + C_1 \exp(\eta) + C_0 + C_{-1} \exp(-\eta) \right] = 0,
\]

where

\[
A = \left[ \exp(\eta) + b_0 + b_{-1} \exp(-\eta) \right]^3.
\]
and the $C_n$ are coefficients of exp$(n \eta )$. Equating to zero the coefficients of all powers of exp$(n \eta )$ yields a set of algebraic equations for $a_0, b_0, a_1, a_{-1}, b_0, k, p, q$ and $c'$. Solving the system of algebraic equations with the aid of Maple 13, we obtain the following.

**Case 1.**

\[ k = k, \quad p = p, \quad q = q, \quad c' = \frac{1}{2} p^2 + \frac{1}{2} q - \frac{1}{2} k^2, \quad a_1 = 0, \quad a_{-1} = 0, \]

\[ b_0 = 0, \quad a_0 = a_0, \quad b_{-1} = - \frac{1}{4} a_0^2 (1 + 4 p^2 - \gamma) \frac{1}{k^2 (2 p - 1)(2 p + 1)} \]

Substituting Eq. (3.36) into (3.33) and inserting the result into the transformations (3.3) and (3.4), we obtain the generalized solitary wave solutions of Eqs. (1.3) and (1.4)

\[ u(\eta) = \frac{a_0}{e^{\eta} - \frac{1}{4} a_0^2 (1 + 4 p^2 - \gamma) \frac{1}{k^2 (2 p - 1)(2 p + 1)} e^{-\eta}} \]

\[ \psi(x,t) = \left[ \frac{a_0}{e^{\eta} - \frac{1}{4} a_0^2 (1 + 4 p^2 - \gamma) \frac{1}{k^2 (2 p - 1)(2 p + 1)} e^{-\eta}} \right] e^{i(px+qt)} \]

and

\[ v(x,t) = \frac{u^2}{1 - 4 p^2} + c' = \frac{u^2(\eta)}{1 - 4 p^2} + \left( \frac{1}{2} p^2 + \frac{1}{2} q - \frac{1}{2} k^2 \right) \]

where $\eta = k(x - 2 pt)$ and $a_0$ is an arbitrary parameter which can be determined by initial and boundary conditions.

The case in which $k$ is an imaginary number, the obtained solitary solution (3.37) can be converted into periodic solution [15], we write $k = iK$ where $K$ is a real number.

Using the transformation

\[ \eta = i \zeta, \quad \zeta = K(x - 2 pt), \]

\[ \exp(\eta) = \cos(\zeta) + i \sin(\zeta), \]

\[ \exp(-\eta) = \cos(\zeta) - i \sin(\zeta). \]

and substituting Eq.(3.40) into (3.37) yields

\[ u(\zeta) = \frac{a_0}{(1 + m) \cos(\zeta) + (1 - m) i \sin(\zeta)}, \]
where \( m = \frac{1}{4} \frac{a_0^2 (1 + 4 \gamma p^2 - \gamma)}{K^2 (2p - 1)(2p + 1)} \).

If we look for a periodic or a compact-like solution, the imaginary part in the denominator of Eq. (3.41) must vanish, which requires

\[
1 - m = 1 - \frac{1}{4} \frac{a_0^2 (1 + 4 \gamma p^2 - \gamma)}{K^2 (2p - 1)(2p + 1)} = 0,
\]

(3.42)

Solving \( a_0 \) from Eq. (3.42) we obtain

\[
a_0 = \pm 2K \frac{(2p - 1)(2p + 1)}{\sqrt{1 + 4 \gamma p^2 - \gamma}}.
\]

(3.43)

Substituting Eq. (3.43) into Eq. (3.41) results in a periodic solution, which reads

\[
u(x, t) = u(\zeta) = \pm K \frac{(2p - 1)(2p + 1)}{\sqrt{1 + 4 \gamma p^2 - \gamma}} \sec(\zeta),
\]

(3.44)

and

\[
\psi(x, t) = \pm K \frac{(2p - 1)(2p + 1)}{\sqrt{1 + 4 \gamma p^2 - \gamma}} \sec^2(\zeta) e^{i(px + qt)},
\]

(3.45)

\[
\nu(x, t) = -\frac{K^2}{2(1 + 4 \gamma p^2 - \gamma)} \sec^2(\zeta) + c',
\]

(3.46)

where \( \zeta = K(x - 2pt) \) and \( c' = \frac{1}{2} p^2 + \frac{1}{2} q + \frac{1}{2} K^2 \).

Case 2.

\[
k = k, \quad p = \pm \frac{1}{2} \sqrt{\frac{q - 1}{\gamma}}, \quad q = q, \quad b_{-1} = 0, \quad a_1 = 0, \quad a_{-1} = a_0 b_0,
\]

(3.47)

\[
b_0 = b_0, \quad a_0 = a_0, \quad c' = -\frac{1}{8} \left( \frac{-4 \gamma q + 4 \gamma k^2 + 1 - \gamma}{\gamma} \right).
\]

By the same manipulation as illustrated in the previous case, we can finally obtain the generalized solitary wave solutions of Eqs. (1.3) and (1.4) as

\[
u(\zeta) = \frac{a_0 (1 + b_0 e^{-\eta})}{e^{\eta} + b_0},
\]

(3.48)
and

$$\psi(x, t) = \left[ a_0 (1 + b_0 e^{-\eta}) \right] e^{\frac{1}{2} \sqrt{\frac{\gamma - 1}{\gamma} t}} e^{i \sqrt{\frac{\gamma - 1}{\gamma} (x + q \eta)}}$$

(3.49)

where $\eta = k \left( x \pm \sqrt{\frac{\gamma - 1}{\gamma} t} \right)$ and $\frac{\gamma - 1}{\gamma} > 0$,

$$v(x, t) = \frac{u^2(\eta)}{1 - 4 p^2} + c' = \gamma u^2 - \frac{1}{8} \left( -4pq + 4k^2 + 1 - \gamma \right),$$

(3.50)

where $a_0$ and $b_0$ are free parameters; for example, if we set $b_0 = \pm 1$ in Eq. (3.48), the solution reduces to

$$u(\eta) = a_0 [\cosh(\eta) - \sinh(\eta)].$$

(3.51)

Case 3.

$$k = \pm \frac{a_{-1}}{b_{-1}} \sqrt{\gamma + \frac{1}{4p^2 - 1}}, \quad p = p, \quad q = q, \quad a_{-1} = a_{-1}, \quad a_1 = -\frac{a_{-1}}{b_{-1}}, \quad a_0 = 0,$$

$$b_0 = 0, \quad b_{-1} = b_{-1}, \quad c' = \frac{a_{-1}^2}{b_{-1} (4p^2 - 1)} + \frac{1}{2} \left( p^2 + q + 2 \gamma \left( \frac{a_{-1}}{b_{-1}} \right)^2 \right)$$

(3.52)

and consequently we get

$$u(\eta) = \frac{a_{-1} (-e^\eta + b_{-1} e^{-\eta})}{b_{-1} (e^\eta + b_{-1} e^{-\eta})},$$

(3.53)

where $\eta = \pm \frac{a_{-1}}{b_{-1}} \sqrt{\gamma + \frac{1}{4p^2 - 1} (x - 2pt)}$ and $\gamma + \frac{1}{4p^2 - 1} > 0$,

and $a_{-1}, \quad b_{-1}$ are free parameters; for example, if we put $b_{-1} = 1$ in (3.53), our general solutions reduce to

$$u(\eta) = -a_{-1} \tanh(\eta),$$

(3.54)

$$\psi(x, t) = -a_{-1} \tanh(\eta) e^{i(px + q\eta)},$$

(3.55)
\[ \nu(x,t) = \frac{u^2}{1 - 4p^2} + c' = a_{-1}^2 \tanh^2(\eta) + \left[ \frac{a_{-1}^2}{(4p^2 - 1)} + \frac{1}{2} \left( p^2 + q + 2\gamma a_{-1}^2 \right) \right]. \tag{3.56} \]

If we take \( b_{-1} = -1 \) in solution (3.53), we obtain

\[ u(\eta) = a_{-1} \coth(\eta), \tag{3.57} \]

\[ \psi(x,t) = a_{-1} \coth(\eta)e^{i(px+qy)}, \tag{3.58} \]

\[ \nu(x,t) = \frac{u^2}{1 - 4p^2} + c' = a_{-1}^2 \coth^2(\eta) + \left[ \frac{a_{-1}^2}{(4p^2 - 1)} + \frac{1}{2} \left( p^2 + q + 2\gamma a_{-1}^2 \right) \right]. \tag{3.59} \]

If we set \( a_{-1} = -\frac{\alpha \sqrt{\lambda^2 - 4\mu}}{\lambda} \) in (3.54)-(3.56) and recall \( q \) from Eq. (3.9); also, if we put \( a_{-1} = -\frac{\alpha \sqrt{\lambda^2 - 4\mu}}{\lambda} \) in (3.57)-(3.59) and recall \( q \) from Eq. (3.9), then it can be easily converted to the same solutions (3.14)-(3.17), respectively.

**Case 4.**

\[ k = \pm 8 \frac{a_{-1}}{b_0^2} \sqrt{\gamma + \frac{1}{4p^2 - 1}}, \quad p = p, \quad q = q, \quad a_{-1} = a_{-1}, \quad a_1 = -\frac{4a_{-1}}{b_0^2}, \quad a_0 = 0, \tag{3.60} \]

\[ b_0 = b_0, \quad b_{-1} = \frac{b_0^2}{4}, \quad c' = \frac{16a_{-1}}{b_0^2(4p^2 - 1)} + \frac{1}{2} \left( p^2 + q + \frac{32\gamma a_{-1}^2}{b_0^4} \right). \]

By the same procedure as illustrated above, we can finally obtain the following exact solutions:

\[ u(\eta) = \frac{-4a_{-1}e^\eta + a_{-1}e^{-\eta}}{e^\eta + b_0 + \frac{b_0^2}{4}e^{-\eta}}, \tag{3.61} \]

where \( \eta = \pm 8 \frac{a_{-1}}{b_0^2} \sqrt{\gamma + \frac{1}{4p^2 - 1}}(x - 2pt) \) and \( \gamma + \frac{1}{4p^2 - 1} > 0 \).

and \( a_{-1}, \quad b_0 \) are free parameters; for example, if we set \( b_0 = 2 \) in the solution (3.61), we get

\[ u(\eta) = -a_{-1} \tanh\left( \frac{\eta}{2} \right), \tag{3.62} \]
\[ \psi(x,t) = -a_{-1} \tanh \left( \frac{\eta}{2} \right) e^{i(px+qt)} \]  
(3.63)

\[ \nu(x,t) = \frac{u^2}{1 - 4p^2} + c' = \frac{a_{-1}^2 \tanh^2 \left( \frac{\eta}{2} \right)}{1 - 4p^2} + \left[ \frac{a_{-1}^2}{(4p^2 - 1)} + \frac{1}{2} \left( p^2 + q + 2\alpha^2 \right) \right]. \]  
(3.64)

If we put \( b_0 = -2 \) in the solution (3.61), we have

\[ u(\eta) = -a_{-1} \coth \left( \frac{\eta}{2} \right). \]  
(3.65)

If we set \( a_{-1} = -\frac{\alpha_0 \sqrt{\lambda^2 - 4\mu}}{\lambda} \) in (3.62)-(3.65) and recall \( q \) from Eq. (3.9), then it can be easily converted to the same solutions (3.14)-(3.17), respectively.

**Remark 1.** The particular cases (3.14)-(3.17), (3.54)-(3.59) and (3.62)-(3.65) of the general solutions obtained by the proposed methods have been compared and verified together.

**Remark 2.** We have verified all the obtained solutions by putting them back into the original equations (1.3) and (1.4) with the aid of Maple 13.

### 4. Conclusions

To sum up, the purpose of the study is to show that exact solutions of a nonlinear Schrödinger-type system known as the generalized Zakharov equations can be obtained by the \((G'/G)\)-expansion method and the Exp-function method. Some of the particular cases of the obtained general solutions by the proposed methods have been compared and verified together. New and more general exact solutions, not obtained by the previously available methods, are also found. It can be seen that the Exp-function method yields more general solutions in comparison with the other method. Overall, the results reveal that the \((G'/G)\)-expansion and Exp-function methods are powerful mathematical tools to solve nonlinear partial differential equations (NPDEs) in the terms of accuracy and efficiency. This is important, since systems of NPDEs have many applications in engineering.

### References


An adaptive spectral variational iteration method for solving nonlinear initial value problems

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Abstract

The successive iterations of the variational iteration method (VIM) in solving nonlinear initial value problems (NIVPs) may be very complex so that the resulting integrals in its iterative relation may not be performed analytically. In order to completely eliminate this restriction, in this paper, an adaptive spectral VIM is proposed for solving the NIVPs. The obtained results here demonstrate excellent performance of this algorithm.

Keywords: Variational iteration method; Spectral collocation; Nonlinear initial value problems

1 Introduction

In many circumstances, nonlinear ordinary differential equations will model the dynamical behaviour of certain mechanical systems, whereby exact solutions or a closed form of analytical solutions are very difficult to obtain. In general, we have to rely on numerical integration, some particular transformations, linearization or discretization in order to obtain their approximate solutions. Also, there has been much attention devoted to search for better and more efficient methods for determining solutions, approximate or exact, analytical or numerical, to these kinds of nonlinear equations (see, e.g., [1-4] and the references therein).

The variational iteration method (VIM) plays an important role in recent researches. This method is proposed by He [5-10] as a modification of a general Lagrange multiplier method [11]. It has been shown that this procedure is a powerful tool for solving various kinds of problems (e.g., see [12-15]). We believe that an easy-to-use algorithm can be proposed for solving the NIVPs. Therefore, the strategy that will be pursued in this work rests mainly on establishing an effective algorithm based on the VIM and the spectral collocation scheme (e.g., see [16]) for obtaining a highly accurate approximate solution of the NIVPs. The example analyzed in this paper shows that the developed algorithm is very effective to solve the NIVPs as compared to the Matlab ode45 solver.

2 Legendre approximation

This section is devoted to introducing Legendre functions and expressing some basic properties of them. Let \( P_i(t) \) be the standard Legendre polynomial of degree \( i \). The shifted Legendre polynomials to \( \Lambda = [0, T] \), \( P_{t,j}(t) \), are defined by
\[ P_{r,i}(t) = P\left(\frac{2t}{T} - 1\right) = \frac{(-1)^i}{i!} \frac{d^i}{dt^i} \left(t^i \left(1 - \frac{t}{T}\right)^i\right), \quad i = 0,1,2,..., \tag{1} \]

and it satisfies the recursive relation
\[ P_{r,i+1}(t) = \frac{2i+1}{i+1} \left(\frac{2t}{T} - 1\right) P_{r,i}(t) - \frac{i}{i+1} P_{r,i-1}(t), \quad i \geq 1. \tag{2} \]

The set of the shifted Legendre polynomials \( P_{r,i}(t) \) is a complete \( L^2(\Lambda) \)-orthogonal system in \( \Lambda \), i.e.,
\[ \int_{\Lambda} P_{r,i}(t) P_{r,j}(t) dt = \frac{T}{2i+1} \delta_{i,j}, \tag{3} \]
where \( \delta_{i,j} \) is the Kronecker symbol. Thus for any \( v \in L^2(\Lambda) \), we have that
\[ v(t) = \sum_{i=0}^{\infty} v_{r,i} P_{r,i}(t), \quad v_{r,i} = \frac{2i+1}{T} \int_{\Lambda} v(t) P_{r,i}(t) dt. \tag{4} \]

We denote by \( t_j, 0 \leq j \leq N \), the nodes of the standard Legendre-Gauss-Lobatto (LGL) interpolation on the interval \([-1,1]\). The corresponding weights are \( w_j, 0 \leq j \leq N \). The nodes of the shifted LGL interpolation on \( \Lambda \) are denoted by \( t_{r,j} = \frac{T}{2}(t_j + 1), 0 \leq j \leq N \). The corresponding weights are \( w_{r,j} = \frac{T}{2} w_j, 0 \leq j \leq N \). Let \( \pi_N(\Lambda) \) be the set of polynomials of degree at most \( N \). Also, let \( (u,v)_T \) and \( \|v\|_T \) be the inner product and the norm of space \( L^2(\Lambda) \), respectively. The shifted LGL interpolation \( I_{T,N}v(t) \in \pi_N(\Lambda) \) is determined by
\[ I_{T,N}v(t_{r,j}) = v(t_{r,j}), \quad 0 \leq j \leq N. \tag{5} \]

We can expand \( I_{T,N}v(t) \) as
\[ I_{T,N}v(t) = \sum_{i=0}^{N} v_{r,i} P_{r,i}(t), \tag{6} \]
where
\[ v_{r,i} = \frac{2i+1}{T} (I_{T,N}v, P_{r,i})_T = \frac{2i+1}{T} (v, P_{r,i})_T, \quad 0 \leq i \leq N. \tag{7} \]

Now, let us consider a model IVP as follows (with this assumption that the problem has the unique solution on the interval \( \Lambda \)):
\[
\begin{align*}
\frac{d}{dt} u(t) &= f(t,u), \quad t \in \Lambda, \\
u(0) &= u_0,
\end{align*}
\]

where \( \frac{d}{dt} u(t) \) and \( f \) are assumed to be continuous in \( \Lambda \). The Legendre pseudospectral method based on the LGL points for solving (8) is to seek \( u^N(t) \in \pi_N(\Lambda) \) such that

\[
\begin{align*}
\frac{d}{dt} u^N(t_{r,k}) &= f(t_{r,k}, u^N(t_{r,k})), \quad 0 \leq k \leq N, \\
u^N(0) &= u_0.
\end{align*}
\]

Now we need an iterative method to solve the problem (9). In this work, we shall apply the explicit VIM in an adaptive manner.

### 3 The spectral variational iteration method

The idea of the VIM is very simple and straightforward. To explain the basic idea of the VIM, first we consider Eq. (9) as:

\[
L[u(t)] + N[u(t)] = g(t),
\]

where \( L \) with the property \( Lv \equiv 0 \) when \( v \equiv 0 \) denotes the linear operator with respect to \( u \), \( N \) is a nonlinear operator with respect to \( u \) and \( g(t) \) is the source term. According to [15], we then construct the following explicit VIM for Eq. (10):

\[
L[u_{n+1}(t)] - u_n(t) = -A[u_n(t)],
\]

with the initial condition

\[
u_{n+1}(0) = u_0, \quad \forall n,
\]

where \( u_0(t) \) is the initial guess (the initial guess can be freely found from solving its corresponding linear equation \( L[u_0(t)] = 0 \) or \( L[u_0(t)] = g(t) \)) and the subscript \( n \) denotes the \( n \)-th iteration, and

\[
A[u_n(t)] = L[u_n(t)] + N[u_n(t)] - g(t) \equiv \frac{d}{dt} u_n(t) - f(t, u_n(t)).
\]

Accordingly, the successive approximations \( u_n(t), n \geq 0 \) of the VIM will be readily obtained by choosing all the above-mentioned parameters. Consequently, the exact solution may be obtained by using
Now we are in a position to construct the numerical algorithm based on the VIM for (9). To do this, in view of (11) and (12), we derive the following stable scheme for solving (9), which is called the spectral VIM,

\[
\begin{cases}
L[u_n^{N}(t_{T,k}) - u_n^{N}(t_{T,k})] = -A[u_n^{N}(t_{T,k})], & 0 \leq k \leq N, \\
u_n^{N}(0) = u_0, & n \geq 0.
\end{cases}
\]

As pointed out before, this is an explicit approach and under certain conditions has a unique solution. Here, in order to directly calculate the unknown \(u_n^{N}(t_{T,k})\), we give a simple implementation by expanding \(u_n^{N}(t_{T,k})\) by the shifted Legendre polynomials, which leads to a stable algorithm. The polynomial \(p(t) \approx \sum_{j=0}^{N} u_j^{N} P_{T,j}(t)\), interpolates the points \((t_{T,j}, u_j^{N})\), \(0 \leq j \leq N\), that is, \(p(t_{T}) = u_n^{N}\). The value of the interpolating polynomial's \(r\) th derivative at the nodes is \(p^{(r)}(t_{T}) = D^{(r)}u_n^{N}\). These differentiation matrices are studied in [16,17] where also explicit expressions for them are given. Generally, in order to solve (9) using (15), the interpolating polynomial \(p(t)\) is required to satisfy the equation at the interior nodes \((m = 1 : N)\), i.e., \(p(t_{T,m}) = (u_n^{N})_m = I_m u_n^{N}\) where \(I_m\) denotes the \(m\) row of the \((N+1) \times (N+1)\) identity matrix, and the derivative value is \(p^{(r)}(t_{T,m}) = D^{(r)}u_n^{N}\) where \(D^{(r)}\) denotes the \(m\) row of the differentiation matrix \(D^{(r)}\). The initial condition that involves the value of the interpolating polynomial can be handled by using the formula \(p(t_{T,0}) = (u_n^{N})_0 = I_0 u_n^{N}\) where \(I_0\) denotes the first row of the identity matrix. Substituting the above matrix relations, the solution vector \(u_n^{N}\) can be found by solving the following matrix equation:

\[
L u_n^{N} = L u_0^{N} - A[u_n^{N}],
\]

where the matrix \(L\) and the vector \(A[u_n^{N}]\) are as below:

\[
L = \begin{bmatrix}
I_0; \\
L_m;
\end{bmatrix}, \quad A[u_n^{N}] = \begin{bmatrix}
I_0; u_n^{N} - u_0; \\
L_m; u_n^{N} + \text{diag}(N[u_n^{N}]) - g(t_{T,m})
\end{bmatrix}.
\]

Now, an easy and reliable way of ensuring the validity of approximations for large \(t\) is to determine the solution in a sequence of intervals of \(t\), which are subject to continuity conditions at the end points of each interval. Thus, define the following set of disjoint intervals \(I_{\Delta s}^s = [t_s, t_{s+1}]\) with \((\Delta t)_s = \Delta_s = t_{s+1} - t_s\), \(s = 0, 1, ..., K - 1\), where \(t_0 = 0\), \(t_K = T\), and \(\bigcup_{s=0}^{K-1} [t_s, t_{s+1}] = [0, T]\). Assume that the points \(t_s, 0, ..., t_s, N\) are the LGL nodes on the interval \(I_{\Delta s}^s\) with \(t_{s,0} = t_s\) and \(t_{s,N} = t_{s+1}\). We, therefore, have the following piecewise-spectral VIM (PSV) for solving (9):

\[
u(t) = \lim_{n \to \infty} u_n(t).
\]
\[ L \mathbf{u}^{N}_{r+1,n+1} = L \mathbf{u}^{N}_{r+1,n} - A_j[\mathbf{u}^{N}_{r+1,n}], \]

where \( L \) is noted as in (17) and \( ([\mathbf{u}^{N}_{r,n}]_N) \) denotes the last component of \( \mathbf{u}^{N}_{r,n} \).

\[
A_j[\mathbf{u}^{N}_{r+1,n}] = \begin{bmatrix}
L_{0n} \mathbf{u}^{N}_{r+1} - ([\mathbf{u}^{N}_{r,n+1}]_N) \\
L_{1n} \mathbf{u}^{N}_{r+1} + \text{diag}(N[\mathbf{u}^{N}_{r+1}]) - g(t_{r,n})
\end{bmatrix}.
\]

Thus, starting from the initial approximation \( \mathbf{u}^{N}_{0}(t) \), we can use the recurrence formula (18) to successively obtain directly \( \mathbf{u}^{N}_{n}(t) \) for \( n \geq 0 \). It should be emphasized that, in a similar way, the PSV algorithm is also applicable to systems of ordinary differential equations.

### 4 An adaptive strategy

In this section, the following adaptive strategy [2] is proposed for the PSV algorithm, which we summarize as APSV. This technique simplifies computation, and saves time and work, as will be observed later in this paper.

Let \( \mathbf{u}^{N}_{n+1} \) be the solution of the PSV formula with the step size \( \Delta_n \) and \( \tilde{\mathbf{u}}^{N}_{n+1} \) the solution with the step size \( \Delta_n/2 \). Taking the difference of \( \mathbf{u}^{N}_{n+1} \) and \( \tilde{\mathbf{u}}^{N}_{n+1} \), the local error estimator of \( \mathbf{u}^{N}_{n+1} \), i.e., \( \text{Est} = \tilde{\mathbf{u}}^{N}_{n+1} - \mathbf{u}^{N}_{n+1} \), is defined. This value is an estimation of the main part of the local discretization error of the method. Additionally, let \( k \) be the dimension of the ODE system, and \( \text{Atol} \) and \( \text{Rtol} \) the user-specified absolute and relative error tolerances. The tolerances occurring in each step are denoted by \( \text{Tol}_i = \text{Atol} + \text{Rtol}[\mathbf{u}^{N}_{n+1}] \), \( i = 1, \ldots, k \).

Taking

\[
\text{err} = \left( \frac{1}{k} \sum_{i=1}^{k} \frac{\text{Est}}{\text{Tol}_i} \right)^2,
\]

as a measure we find an optimal step size \( \Delta_{\text{opt}} \) by comparing \( \text{err} \) to 1. Thus we obtain the optimal step size as \( \Delta_{\text{opt}} = \Delta_n \cdot \text{err}^{-\alpha} \), where for \( \text{err} \leq \text{fac}_{\text{err}} \), we use \( \alpha = \frac{1}{N_{\alpha} + 1} \) and \( N_{\alpha} = \max\{N_{\alpha}, N_{\text{min}}\} \), and for \( \text{err} > \text{fac}_{\text{err}} \), \( \alpha = \frac{1}{N_{\alpha}} \) and \( N_{\alpha} = \min\{N_{\alpha} + 1, N_{\text{max}}\} \). This is, of course, not the best choice for all problems. The new step size

\[
\Delta_{\text{new}} = \Delta_{n+1} = \min\left\{ \text{fac}_{\text{max}}, \max\left\{ \text{fac}_{\text{min}}, \text{fac}_{\text{err}}\left(\frac{1}{\text{err}}\right)^\alpha \right\} \right\},
\]

is obtained by using \( \text{err} \) with \( N_{\alpha+1} \) as order of polynomial, instead of order of consistency. The integration of the growth factors \( \text{fac}_{\text{max}} \) and \( \text{fac}_{\text{min}} \) to the relation (21) prevents for too large step
increase and contribute to the safety of the code. Additionally, using the safety factor $\text{fac}$ makes sure that $\text{err}$ will be accepted in the next step with high probability. The step is accepted, in case that $\text{err} \leq \text{fac}_{\text{err}}$ otherwise it is rejected and then the procedure is redone. In both cases the new solution is computed with $\Delta_{\text{new}}$ as step size.

5 An illustrative example

In order to illustrate the efficiency of the APSV described in this paper, we present one example which will perform in Matlab 7 with double precision in a Toshiba A8 (Windows XP Professional) Intel(R) Core(TM)2 Duo Processor T7200. The forced Duffing equation [18]:

$$x'' = x - cx' - x^3 + A \cos(t),$$

(22)
models the damped motion of a mass on a nonlinear spring driven by a periodic forcing function of amplitude $A$. We take initial conditions $x(0) = 0.5$, $x'(0) = 0$ and solve the equation when $c = 0.25$ and $A = 0.3$. The change of variables $u_1(t) = x(t)$ and $u_2(t) = x'(t)$ transforms the second-order IVP of (22) to the following nonlinear system of the first-order differential equations:

$$\begin{cases}
    u'_1(t) = u_2(t), & u_1(0) = 0.5, \\
    u'_2(t) = u_1(t) - cu_2(t) - u_1^3(t) + A \cos(t), & u_2(0) = 0.
\end{cases}$$

(23)

In order to show the efficiency of the above adaptive mechanism controlling the truncation error, we solve the above system using the above-mentioned APSV algorithm. In the framework of the APSV algorithm, for simplicity, we now put the linear operator $L = u'$, the nonlinear operator $N = f(t,u)$ and the source term $g(t) \equiv 0$. The numerical results can be observed in Table 1.

<table>
<thead>
<tr>
<th>Algorithm</th>
<th>$T$</th>
<th>$Atol$</th>
<th>$Rtol$</th>
<th>No. of steps</th>
<th>CPU time (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>APSV</td>
<td>100</td>
<td>$10^{-13}$</td>
<td>$10^{-13}$</td>
<td>76</td>
<td>0.45</td>
</tr>
<tr>
<td>ODE45</td>
<td>100</td>
<td>$10^{-13}$</td>
<td>$10^{-13}$</td>
<td>47889</td>
<td>1.68</td>
</tr>
<tr>
<td>APSV</td>
<td>10000</td>
<td>$10^{-12}$</td>
<td>$10^{-12}$</td>
<td>502</td>
<td>3.43</td>
</tr>
<tr>
<td>ODE45</td>
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<td>$10^{-12}$</td>
<td>$10^{-12}$</td>
<td>314737</td>
<td>19.80</td>
</tr>
<tr>
<td>APSV</td>
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<td>$10^{-11}$</td>
<td>$10^{-11}$</td>
<td>4615</td>
<td>28.02</td>
</tr>
<tr>
<td>ODE45</td>
<td>10000</td>
<td>$10^{-11}$</td>
<td>$10^{-11}$</td>
<td>1961201</td>
<td>617.13</td>
</tr>
</tbody>
</table>

In Table 1, we list the costed number of steps (labeled as No. of steps) for some different values of $L$, $Atol$ and $Rtol$, and the corresponding costed CPU elapsed time (labeled as CPU time). We can observe, from Table 1, that the APSV algorithm costs both less computational time and very smaller steps than the Matlab ode45 solver with the same tolerances for this specific equation.

6 Conclusion
In this paper, we proposed an adaptive piecewise spectral variational iteration method (APSV). As shown in this paper, the developed technique makes simple computation, and saves time and work. The numerical results demonstrate the efficiency of the suggested scheme. They also show that the proposed method costs both lesser computational time and very smaller steps than the Matlab ode45 solver. Although we only considered a model problem in this study, the developed APSV algorithm is applicable further to many other problems.

References

Solution of time-fractional reaction diffusion equation by using homotopy analysis method

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Abstract

The Homotopy analysis method (HAM) is used to obtain an approximate solution of the nonlinear time fractional reaction - diffusion equation. Convergence of the solution and effects for the method are discussed within comparing the obtained results with exact solution of the corresponding nonlinear partial differential equation, which indicated that the proposed method is very effective and simple. It also suggests that both the Homotopy perturbation method (HPM), Adomian decomposition method (ADM) and variational iteration method (VIM) are special cases of the HAM. The HAM contains a certain auxiliary parameter which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution.

Keywords: Homotopy analysis method, Adomian method, homotopy perturbation method, variational iteration method, auxiliary parameter, fractional calculus

1- Introduction.

Nonlinear partial differential equations (NPDEs) are encountered in such various fields as physics, chemistry, biology, mathematics and engineering. Most nonlinear models of real life problems are still very difficult to solve, either numerically or theoretically. In this paper we consider the nonlinear fractional partial differential equation (fractional reaction - diffusion equation) of general form

\[ D_\alpha^\delta u(x,t) = d \frac{\partial^2 u(x,t)}{\partial x^2} + f(u,x,t) \quad (1.1) \]

Where \( d \) is the diffusion coefficient and \( f(u,x,t) \) is a nonlinear function representing reaction kinetics. It is interesting to observe that for \( f(u,x,t) = 6u(1-u), \) Eq. (1.1) reduces to the time-fractional Fisher equation which was originally proposed by Fisher [2] as a model for the spatial and temporal propagation of a virile gene in an infinite medium. If we set \( f(u,x,t) = u(1-u)(u-\alpha), \) it gives rise to the time-fractional Fitzhugh–Nagumo equation, which is an important nonlinear reaction-diffusion equation and applied to model the transmission of nerve impulses [3,4], also used in biology and the area of population genetics in circuit theory [5]. While \( \alpha = 1, \) the Fitzhugh–Nagumo equation reduces to the real Newell–Whitehead equation.
2- Basic definitions.

In this section, we give some definitions and properties of the fractional calculus. The fractional calculus is a name for the theory of integrals and derivatives of arbitrary order. Various definitions of fractional integration and differentiation are found in [11-14], such as Grunwald-Letnikov's definition, Riemann-Liouville definition, and Caputo's definition and generalized function approach. For the purpose of this paper, the Caputo's definition of the fractional differentiation will be used, taking the advantage of Caputo's approach that the initial conditions for fractional differential equation with Caputo's derivatives take on the traditional form as for integer-order differential equation.

**Definition 2.1** Areal function $h(t)$, $t > 0$, is said to be in the space $C^\mu, \mu \in \mathbb{R}$, if there exists a real number $\mu > \mu$, such that $h(t) = t^\mu h_0(t)$, where $h_0(t) \in C(0, \infty)$, and it is said to be in the space $C^\mu$ if and only if $h(t) \in C^\mu, \mu \in \mathbb{N}$.

**Definition 2.2** The Riemann-Liouville fractional integral operator $J^\alpha$ of order $\alpha \geq 0$, of a function $h \in C^\mu, \mu \geq -1$ is defined as

$$J^\alpha h(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t - \tau)^{\alpha-1} h(\tau) \, d\tau$$

(2.1)

$I(\alpha)$ is the well-known gamma function. Some of the $J^\alpha$ properties of the operator which we will need here are as follows:

1. $J^\alpha J^\beta h(t) = J^{\alpha+\beta} h(t)$,
2. $J^\alpha J^\beta h(t) = J^\beta J^\alpha h(t)$,
3. $J^\alpha t^\lambda = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\lambda+1)} t^{\lambda+\alpha}$.

**Definition 2.3** The fractional derivative $(D^\alpha)$ of $h(t)$ in the Caputo's sense is defined as follows

$$D^\alpha h(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t - \tau)^{n-\alpha-1} h^n(\tau) \, d\tau$$

(2.2)

for $n-1 < \alpha < n, n \in \mathbb{N}, t > 0, h \in C^n_{-1}$

The following are two basic properties of Caputo's fractional Derivative [14]

1. Let $h \in C^n_{-1}, n \in \mathbb{N}$ then $D^\alpha h, 0 \leq \alpha \leq n$ is well defined and $D^\alpha h \in C_{-1}$
2. Let $-1 < \alpha < n, n \in \mathbb{N}$ and $h \in C^\mu, \mu \geq -1$ then

$$(J^\alpha D^\alpha) h(t) = h(t) - \sum_{k=0}^{n-1} \frac{h^{(k)}(t)}{k!} (0^+)^{\alpha-k}$$

(2.3)
3- The Homotopy analysis method (HAM).

The HAM [1] is applied to the nonlinear homogeneous fractional reaction - diffusion equation with initial conditions. Then, the studied differential equation is considered as

\[ N[u(x, t)] = 0, \quad (3.1) \]

Where \( N \) is a nonlinear operator for this problem and \( \mathbf{u}(x, t) \) is an unknown function. By means of the HAM, one first constructs the zero-order deformation equation

\[ (1 - q) \mathcal{L} \phi(x, t, q) - \mathbf{u}_0(x, t) = q \mathcal{H}(x, t) N[\phi(x, t)], \quad (3.2) \]

Where \( q \) is the embedding parameter, \( q \in [0, 1] \), \( \mathcal{H} \neq 0 \) is an auxiliary parameter, \( \mathcal{H}(x, t) \neq 0 \) is an auxiliary function, \( \mathcal{L} \) is an auxiliary linear operator, \( \mathbf{u}_0(x, t) \) is an initial guess. Obviously, when \( q = 0 \) and \( q = 1 \), it holds that

\[ \phi(x, t, 0) = \mathbf{u}_0(x, t), \quad \phi(x, t, 1) = \mathbf{u}(x, t). \quad (3.3) \]

Liao [4-10] expanded \( \phi(x, t, q) \) in Taylor series with respect to the embedding parameter \( q \), as follows:

\[ \phi(x, t, q) = \mathbf{u}_0(x, t) + \sum_{m=1}^{\infty} \mathbf{u}_m(x, t) q^m, \quad (3.4) \]

Where

\[ \mathbf{u}_m(x, t) = \left. \frac{\partial^m \phi(x, t, q)}{\partial q^m} \right|_{q=0}. \quad (3.5) \]

Assume that the auxiliary linear operator, the initial guess, the auxiliary parameter \( \mathcal{H} \) and the auxiliary function \( \mathcal{H}(x, t) \) are selected such that the series \( (3.4) \) is convergent at \( q = 1 \), then we have from \( (3.4) \)

\[ \mathbf{u}(x, t) = \mathbf{u}_0(x, t) + \sum_{m=1}^{\infty} \mathbf{u}_m(x, t). \quad (3.6) \]

Let us define the vector

\[ \mathbf{u}_m(t) = \{u_0(x, t), u_1(x, t), u_2(x, t), \ldots, u_m(x, t)\}. \quad (3.7) \]

Differentiating \( (3.2) \) \( m \) times with respect to \( q \), then setting \( q = 0 \) and dividing by \( \mathcal{L} \), the mth-order deformation equation

\[ \mathcal{L}\left[\mathbf{u}_m(x, t) - x_m \mathbf{u}_{m-1}(x, t) \right] = \mathcal{H}(x, t) \mathcal{R}_m(\mathbf{u}_{m-1}). \quad (3.8) \]

Where
And

\[ \mathcal{X}_m = \begin{cases} 0 & m \leq 1 \\ 1 & m > 1 \end{cases} \]  \quad (3.10)

The mth-order deformation Eq. (3.8) becomes linear and it can be easily solved, especially by means of symbolic computation software such as Mathematica, Maple, Matlab.

4-Time- fractional reaction - diffusion equation of the fisher type.

**Application 1**

Firstly, we begin by Fisher's equation.

\[ D_t^\alpha u = u_{xx} + u(1-u). \]  \quad (4.1)

With initial condition

\[ u_0(x,0) = \lambda. \]  \quad (4.2)

According to the (HAM), we choose the auxiliary operator as

\[ \mathcal{L}[\phi(x,t;y)] = D_t^\alpha \phi(x,t;q). \]  \quad (4.3)

With property \( \mathcal{L}[c] = 0 \) where \( c \) is a constant. We define a nonlinear operator as

\[ N[\phi(x,t;y)] = D_t^\alpha \phi(x,t;q) - \frac{\partial^\alpha \phi(x,t;q)}{\partial x^\alpha} \phi(x,t;q) + (\phi(x,t;q))^2. \]  \quad (4.4)

In order to obey the rule of solution expression and the rule of the coefficient periodicity [6], the auxiliary function can be determined uniquely \( H(x,t) = 1 \), and

\[ R_m(\bar{u}_{m-1}) = D_t^\alpha u_{m-1} - \frac{\partial^\alpha}{\partial x^\alpha} u_{m-1} - u_{m-1} + \sum_{i=0}^{m-2} w_i w_{m-1-i}. \]  \quad (4.5)

Now the solution of the mth-order deformation equations (3.6) for \( m \geq 1 \) becomes

\[ u_m(x,t) = \mathcal{X}_m u_{m-1}(x,t) + \lambda t^{-2} R_m(\bar{u}_{m-1}). \]  \quad (4.6)

So, the first three terms of the solution are

\[ u_0(x,t) = \lambda \]

\[ u_1(x,t) = \frac{(-1 + \lambda) \lambda h t^\alpha}{\Gamma[1 + \alpha]} \]

\[ u_2(x,t) = \frac{(-1 + \lambda) \lambda h (1 + \lambda) t^\alpha}{\Gamma[1 + 2\alpha]} + \frac{(-1 + \lambda) \lambda (-1 + 2\lambda) h^2 t^{2\alpha}}{\Gamma[1 + 2\alpha]} \]
Then, we can conclude that

\[ u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) + \ldots \]

\[
\begin{align*}
\frac{\partial u}{\partial t} & = -\lambda + \frac{(1-\lambda)\lambda t^{2\alpha}}{\Gamma[1+\alpha]} + \frac{(-1+2\lambda)\lambda t^{2\alpha}}{\Gamma[1+2\alpha]} \\
& \quad + \frac{(-1+2\lambda)\lambda t^{2\alpha}}{\Gamma[1+3\alpha]} \frac{1}{1+2(1+\lambda) + 3(1+\lambda)^2 + \ldots} \\
& \quad + \frac{(-1+2\lambda)\lambda t^{2\alpha}}{\Gamma[1+3\alpha]} \frac{1}{\Gamma[1+\alpha]^2 \Gamma[1+3\alpha]} \frac{1}{1+3(1+\lambda) + 6(1+\lambda)^2 + \ldots} + \ldots
\end{align*}
\]

(4.7)

Using the Geometric series as [16], when the series tends to infinity and \( h \) must be less than \( 0 \), the solution becomes independent of \( h \) and takes the following form

\[
\begin{align*}
\frac{\partial u}{\partial t} & = -\lambda + \frac{(1-\lambda)\lambda t^{2\alpha}}{\Gamma[1+\alpha]} + \frac{(-1+2\lambda)\lambda t^{2\alpha}}{\Gamma[1+2\alpha]} \\
& \quad + \frac{(-1+2\lambda)\lambda t^{2\alpha}}{\Gamma[1+3\alpha]} \frac{1}{1+2(1+\lambda) + 3(1+\lambda)^2 + \ldots} \\
& \quad + \frac{(-1+2\lambda)\lambda t^{2\alpha}}{\Gamma[1+3\alpha]} \frac{1}{\Gamma[1+\alpha]^2 \Gamma[1+3\alpha]} \frac{1}{1+3(1+\lambda) + 6(1+\lambda)^2 + \ldots} + \ldots
\end{align*}
\]

At \( \alpha = 1 \) the solution is the same as [9] and [10] which is a closed form with the exact solution

\[ u(x,t) = \frac{\lambda t^\alpha}{1-\lambda t^\alpha} \]

of the equation (4.1).

The numerical values of the solution with different values of \( \alpha \) and \( t \) indeed of the exact solution illustrate in Table [1], see also Fig [1].

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>(HAM) exact solution</th>
<th>(HAM) exact solution</th>
<th>(HAM) exact solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \alpha = 1 )</td>
<td>0.321413893178</td>
<td>0.321413893178</td>
<td>0.428571428571</td>
</tr>
<tr>
<td>( \alpha = 0.75 )</td>
<td>0.349611867950</td>
<td>0.349611867950</td>
<td>0.500000000000</td>
</tr>
<tr>
<td>( \alpha = 0.2 )</td>
<td>0.384551576393</td>
<td>0.384551576393</td>
<td>0.580000000000</td>
</tr>
</tbody>
</table>
Application 2

Consider the Fisher's equation.

\[
D_t^\alpha u = u_{xx} + au(1 - u)
\]  
(4.9)

With initial condition

\[
u_0(x, 0) = \frac{1}{\left(1 + e^{-x}\right)^2}
\]  
(4.10)

Similarly, choosing

\[
N[\phi(x, t, q)] = D_t^\alpha \phi(x, t, q) - \frac{\phi''(x, t, q)}{\phi(x, t, q)} - a\phi(x, t, q) + a(\phi(x, t, q))^2
\]  
(4.11)

Then,

\[
R_m(u_{m-1}) = D_t^\alpha u_{m-1} - \frac{\phi''(x, t, q)}{\phi(x, t, q)} u_{m-1} - au_{m-1} + a \sum_{i=0}^{m-1} u_i u_{m-1-i}
\]  
(4.12)

So, a first three terms of the series are

\[
u_0(x, t) = \frac{1}{\left(1 + e^{-x}\right)^2}
\]

\[
u_1(x, t) = \frac{5ae\beta e^{\beta}}{3(1 + e^{\beta})^3} t \Gamma[1 + \alpha]
\]
Then, we can conclude that

\[
\begin{align*}
u(x, t) &= u_0(x, t) + u_1(x, t) + u_2(x, t) + u_3(x, t) + \cdots \\
u(x, t) &= \frac{1}{(1 + e^{\sqrt{\varepsilon}})^2} \\
&\quad - \frac{5 a e^{\sqrt{\varepsilon}} h^{2+x}}{3(1 + e^{\sqrt{\varepsilon}})^3 \Gamma[1 + \alpha]} \left[1 + (1 + h) + (1 + h)^2 + (1 + h)^3 \ldots \right] \\
&\quad + \frac{25 a^3 (-1 + 2 e^{\sqrt{\varepsilon}}) e^{\sqrt{\varepsilon}} h^{2+2x}}{18(1 + e^{\sqrt{\varepsilon}})^4 \Gamma[1 + 2\alpha]} \left[1 + 2(1 + h) + 3(1 + h)^2 + \cdots \right] \\
&\quad + \frac{25 a^3 e^{\sqrt{\varepsilon}} h^{3+2x}}{108(1 + e^{\sqrt{\varepsilon}})^6 \Gamma[1 + 3\alpha]} \left[-5 + 15 e^{\sqrt{\varepsilon}} \sqrt{\varepsilon} - 20 e^{\sqrt{\varepsilon}} + 6 e^{\sqrt{\varepsilon}} \right] \\
&\quad + \frac{12 a^3 e^{\sqrt{\varepsilon}} \Gamma[1 + 2\alpha]}{\Gamma[1 + \alpha]^2} \left[1 + 3(1 + h) + 6(1 + h)^2 + \cdots \right] + \cdots
\end{align*}
\]

As the series tends to infinity (using Geometric series as [16] where \( h \) must be less than \( \theta \)), the solution becomes independent of \( h \) in the following form
At $\alpha = 1$ the solution is the same as [9] and [10] which is a closed form with the exact solution
\[ u(x,t) = \frac{1}{(1 + e^{\sqrt{\beta} \cdot t})^2} \]
Of the equation (4.9).

At $\alpha = -1$ the solution is the same as [7] and when $\alpha = 6$ the solution is the same as [8].

Table [2] illustrates the numerical values of the solution with different values of $\alpha$, see also Fig [2], [3].

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta = 1$</th>
<th>$\beta = 0.8$</th>
<th>$\beta = 0.75$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>(HAM)</td>
<td>(HAM)</td>
<td>(HAM)</td>
</tr>
<tr>
<td>0.2</td>
<td>0.0169282153</td>
<td>0.0169279961</td>
<td>0.0213871350</td>
</tr>
<tr>
<td>0.4</td>
<td>0.0201179246</td>
<td>0.0201172756</td>
<td>0.0297186428</td>
</tr>
<tr>
<td>0.6</td>
<td>0.0238595182</td>
<td>0.0238370510</td>
<td>0.0391872610</td>
</tr>
<tr>
<td>0.8</td>
<td>0.0281178229</td>
<td>0.0281148155</td>
<td>0.0509135817</td>
</tr>
</tbody>
</table>

Fig [2] exact solution and approximate solution ($\alpha = 0.75$)

Fig [3] exact solution and approximate solution ($\alpha = 1$)

5- Nonlinear diffusion equation of the Fisher type.

The equation has a form
With initial condition
\[ u_0(x,0) = \frac{1}{1 + \frac{x}{\alpha}} \]  
(5.2)

Similarly, on choosing
\[ N[\phi(x, t, q)] = D_\phi^\xi \phi(x, t, q) - \frac{\partial^2 \phi(x, t, q)}{\partial x^2} + (\phi(x, t, q))^3 - (1 + \alpha) (\phi(x, t, q))^2 + \alpha \phi(x, t, q) \]

Then
\[ R_{n-1}(\tilde{u}_{n-1}) = D_\phi^\xi u_{n-1} - \frac{\partial^2}{\partial x^2} u_{n-1} + \sum_{t=0}^{m-1} \sum_{j=0}^{m-1} u_{t-j} u_{n-1-t-j} - (1 + \alpha) \sum_{t=0}^{m-1} u_{t} u_{n-1-t} + \alpha u_{n-1} \]  
(5.3)

So, a first three terms of series are as follows
\[ u_0(x, t) = \frac{1}{1 + \frac{x}{\alpha}} \]
\[ u_1(x, t) = \frac{(-1 + 2\alpha) e^{\frac{\beta}{\alpha} h^2 t^2}}{2(1 + \frac{x}{\alpha})^2 \Gamma[1 + \alpha]} \]
\[ u_2(x, t) = \frac{(-1 + 2\alpha) e^{\frac{\beta}{\alpha} h(1 + h) t^2}}{2(1 + \frac{x}{\alpha})^2 \Gamma[1 + \alpha]} + \frac{(-1 + 2\alpha)^2 (1 - e^{\frac{\beta}{\alpha} h^2}) e^{\frac{\beta}{\alpha} h^2 t^2}}{4(1 + \frac{x}{\alpha})^2 \Gamma[1 + 2\alpha]} \]
\[ u_3(x, t) = \frac{(-1 + 2\alpha) e^{\frac{\beta}{\alpha} h(1 + h) t^2}}{2(1 + \frac{x}{\alpha})^2 \Gamma[1 + \alpha]} - \frac{32(-1 + 2\alpha)^2 e^{2h h^2 (1 + h)} t^{2\alpha}}{(-1 + \frac{x}{\alpha})^3 (1 + \frac{x}{\alpha})^2 \Gamma[1 + 2\alpha]} \]
\[ + \frac{h^3 t^{3\alpha}}{4(1 + \frac{x}{\alpha})^3 \Gamma[1 + \alpha] \Gamma[1 + 3\alpha]} \]
Then, we can conclude that

\[ u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + u_4(x,t) \ldots \]

Using the Geometric series as [16] such that \( \hat{h} \) must be less than \( \mathbf{U} \), when the series tends to infinity, the solution becomes independent of \( \hat{h} \) and takes the following form

\[
u(x,t) = \frac{1}{1 + e^{\frac{x}{\sqrt{\hat{h}}}}} \left\{ \frac{(-1 + 2a)e^{\frac{x}{\sqrt{\hat{h}}}}}{2(1 + e^{\frac{x}{\sqrt{\hat{h}}}})^2 \Gamma[1 + \alpha]} [1 + (1 + h) + (1 + h)^2 + (1 + h)^3 + \ldots] \right. \\
+ \left. \frac{(-1 + 2a)^2(1 - e^{\frac{x}{\sqrt{\hat{h}}}})}{4(1 + e^{\frac{x}{\sqrt{\hat{h}}}})^3 \Gamma[1 + 2\alpha]} [1 + 2(1 + h) + 3(1 + h)^2 + \ldots] \right. \\
+ \left. \frac{(-1 + 2a)^3e^{\frac{x}{\sqrt{\hat{h}}}}(1 + \alpha + (-2 + a)e^{\frac{x}{\sqrt{\hat{h}}}})\Gamma[1 + 2\alpha]}{4(1 + e^{\frac{x}{\sqrt{\hat{h}}}})^3 \Gamma[1 + 3\alpha]} [1 + 3(1 + h) + 6(1 + h)^2 + \ldots] + \cdots \right\} + \cdots \]  \hspace{1cm} (5.4)

Using the Geometric series as [16] such that \( \hat{h} \) must be less than \( \mathbf{U} \), when the series tends to infinity, the solution becomes independent of \( \hat{h} \) and takes the following form

\[
u(x,t) = \frac{1}{1 + e^{\frac{x}{\sqrt{\hat{h}}}}} \left\{ \frac{(-1 + 2a)e^{\frac{x}{\sqrt{\hat{h}}}}}{2(1 + e^{\frac{x}{\sqrt{\hat{h}}}})^2 \Gamma[1 + \alpha]} + \frac{(-1 + 2a)^2(1 - e^{\frac{x}{\sqrt{\hat{h}}}})}{4(1 + e^{\frac{x}{\sqrt{\hat{h}}}})^3 \Gamma[1 + 2\alpha]} \right. \\
+ \left. \frac{(-1 + 2a)^3e^{\frac{x}{\sqrt{\hat{h}}}}(1 + \alpha + (-2 + a)e^{\frac{x}{\sqrt{\hat{h}}}})\Gamma[1 + 2\alpha]}{4(1 + e^{\frac{x}{\sqrt{\hat{h}}}})^3 \Gamma[1 + 3\alpha]} \right\} + \cdots \]  \hspace{1cm} (5.5)

At \( \alpha = 1 \) the solution is the same as [9] and [10] which is a closed form with the exact solution \( u(x,t) = \frac{1}{1 + e^{\frac{x}{\sqrt{\hat{h}}}} - \frac{x}{\sqrt{\hat{h}}}} \) of the equation (5.1).

At \( \hat{h} = -1 \) the solution is the same with [7], [8].

Table [3] illustrates the numerical values of the solution with different values of \( \alpha, \hat{\xi} \) also see Fig [4], [5].
6- The generalized Fisher equation.

The general form of the studied equation has the form

$$D_t^\alpha u = u_{xx} + u(1 - u^\delta)$$  \hspace{1cm} (6.1)

With initial condition

$$u(x, 0) = \frac{1}{\left(1 + e^{\frac{-x}{\delta}}\right)}$$  \hspace{1cm} (6.2)

Similarly, choose

$$N[\phi(x, t, q)] = D_t^\alpha \phi(x, t, q) - \frac{\partial^2 \phi(x, t, q)}{\partial x^2} - \phi(x, t, q) + \left(\phi(x, t, q)\right)^\delta$$  \hspace{1cm} (6.3)

And

$$R_m \left(\left[\begin{array}{c} u_m-1 \\
\end{array}\right]\right) = D_t^\alpha u_{m-1} - \frac{\partial^2}{\partial x^2} u_{m-1} - u_{m-1}$$

$$+ \sum_{i=0}^{m-1} \sum_{j=0}^{i} \sum_{k=0}^{j} \sum_{r=0}^{k} \sum_{q=0}^{l} u_t u_{x-r} u_{q-r} u_{x-k} u_{l-r} u_{x-m-1-t}$$  \hspace{1cm} (6.4)

So, a first three terms of solution are as follows.
Then, we can conclude that

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \cdots$$

$$u(x,t) = \frac{1}{(1 + e^{3x/2})^{1/3}} + \frac{5e^{3x/2}h(-1 + h)^2t^6}{4(1 + e^{3x/2})^{4/3}(1 + \alpha)} + \frac{25e^{3x/2}(-2 + e^{3x/2}h^2t^2\alpha}{16(1 + e^{3x/2})^{7/3}(1 + 2\alpha)}$$

$$+ \frac{5e^{3x/2}h^3t^{2\alpha}}{64(1 + e^{3x/2})^{15/3}(1 + 3\alpha)} + \frac{105e^{3x/2}h(1 + h)^2(1 + h)t^{2\alpha}}{16(1 + e^{3x/2})^{15/3}(1 + \alpha)^2(1 + 3\alpha)} + \cdots$$

As the series tends to infinity (using Geometric series as [16] where $h$ must be less than 0),

the solution becomes independent of $h$ in the following form

$$u(x,t) = \frac{1}{(1 + e^{3x/2})^{1/3}} + \frac{5e^{3x/2}h^2t^2\alpha}{4(1 + e^{3x/2})^{4/3}(1 + \alpha)} + \frac{25e^{3x/2}(-3 + e^{3x/2}h^2t^2\alpha}{16(1 + e^{3x/2})^{7/3}(1 + 2\alpha)}$$

$$+ \frac{5e^{3x/2}h^3t^{2\alpha}}{64(1 + e^{3x/2})^{15/3}(1 + 3\alpha)} + \frac{105e^{3x/2}h(1 + h)^2(1 + h)t^{2\alpha}}{16(1 + e^{3x/2})^{15/3}(1 + \alpha)^2(1 + 3\alpha)} + \cdots$$
At $\alpha = 1$ the solution is the same as [9] and [10] which is a closed form with the exact solution $u(x,t) = \left(\frac{1}{a} \tan h \left(\frac{-3}{4} \left(x - \frac{3t}{4}\right)\right) + \frac{3}{4}\right)^{1/3}$ of the equation (6.1).

At $\alpha = -1$ the solution is the same with [7].

Table [4] illustrates the numerical values of the solution with different values of $\alpha$ and $t$.

<table>
<thead>
<tr>
<th>$h = -1$, $x = 0.5$</th>
<th>$\alpha = 1$</th>
<th>$\alpha = 0.0$</th>
<th>$\alpha = 0.75$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>(HAM)</td>
<td>(HAM)</td>
<td>(HAM)</td>
</tr>
<tr>
<td>0.1</td>
<td>0.741261805711</td>
<td>0.741281115566193</td>
<td>0.6903649786058341</td>
</tr>
<tr>
<td>0.15</td>
<td>0.766156521022</td>
<td>0.7661831275698736</td>
<td>0.7271702933572694</td>
</tr>
<tr>
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<td>0.799700259898</td>
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<td>0.739838587842454</td>
</tr>
<tr>
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<td>0.817698360209</td>
<td>0.817666436033954</td>
<td>0.766673898077074</td>
</tr>
</tbody>
</table>

Fig [6] exact solution and approximate solution at ($\alpha = 0.75$)

Fig [7] exact solution and approximate solution ($\alpha = 1$)

**Conclusion**

In this paper, the Homotopy analysis method (HAM) is applied to obtain the solution of time-fractional reaction diffusion equation. The results show that (HAM) is powerful and efficient techniques in finding exact and approximate solutions for nonlinear fractional partial differential equations.

The (HAM) provides us with a convenient way to control the convergence of approximation series which is a fundamental qualitative difference in analysis between (HAM) and other method. Thus the auxiliary parameter $h$ plays an important role within the frame of the (HAM). Mathematica has been used for computations in this paper.

**References**


