A new iteration method based on Newton’s method for solving a system of nonlinear equations
K. Sayevand, M. Fardi

Local fractional calculus and fractional complex transform
Xiao-Jun Yang, Ji-Huan He, Dumitru Baleanu

Double-diffusive convection from a permeable vertical surface under convective boundary condition in the presence of heat generation and thermal radiation
P.O. Olanrewaju, F. I. Alao, A. Adeniyan and S.A. Bishop
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A new iteration method based on Newton’s method for solving a system of nonlinear equations

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Abstract

In this letter, we suggest and analyze a new two-step iterative method on the basis of Newton’s method for the approximate solution of the system of nonlinear equations. To illustrate the performance and efficiency of proposed method, several numerical examples are presented. Finally, comparison are made to confirm the effectiveness of our new method.

Keywords: Nonlinear System, Iteration Method, Order of Convergence

1. Introduction

In recent years, some iterative methods for solving the system of nonlinear equations have been presented [1-9]. As we know, the Newton’s method is one of the best iterative methods for solving of nonlinear equations [10]. In this paper, we propose an iterative method on the basis of the Newton’s method. Suppose that we wish to solve a system of nonlinear equations of the form:

\[ F(x) = 0, \]

where \( F(x) : \mathbb{R}^N \rightarrow \mathbb{R}^N \), be a sufficiently differentiable function.

Algorithm (A two step method with cubic convergence): Let \( \bar{X} \in \mathbb{R}^N \) be an estimation of zero of this nonlinear system, the nonlinear system can be written as follows:

\[ F(\bar{X}) + \left( \frac{F'(\bar{X}) + F'(X)}{2} \right)(X - \bar{X}) + G(X) = 0, \]

where

\[ G(X) = F(X) - F(\bar{X}) - \left( \frac{F'(\bar{X}) + F'(X)}{2} \right)(X - \bar{X}). \]

From (1) and (2) we have:

\[ X = \bar{X} - 2[F'(\bar{X}) + F'(X)]^{-1}F(X). \]

By Eq. (3), we suggest the following iterative method on the basis of Newton’s method:

For given \( X_0 = (x_{1,0}, x_{2,0}, \ldots, x_{n,0}) \) approximate the solution \( X_{n+1} \) by the following iterative method:
\[
\begin{align*}
Y_n &= X_n - F(X_n)[F'(X)]^{-1} \\
X_{n+1} &= Y_n - 2[F'(Y_n) + F'(X_n)]^{-1}F(Y_n)
\end{align*}
\] (4)

which is a two-step method.

2. Convergence analysis

Hereunder, the convergence analysis of the family defined by (4) is studied by using the following properties.

**Definition 2.1.** (see [10]) Convergence of order \( P \). Let \( \{X_k\}^\infty_{k=0} \) be a sequence in \( \mathbb{R}^N \), one says that \( \{X_k\}^\infty_{k=0} \) converges to \( W \in \mathbb{R}^N \), with (at least) order \( P \geq 1 \) if

\[
\lim_{k \to \infty} \left( \frac{\|X_{k+1} - W\|}{\|X_k - W\|^P} \right) \neq 0.
\]

**Definition 2.2.** (see [10]) (Computational Order of Convergence). The computational order of convergence (COC) of a sequence \( \{x_n\} \) is defined by

\[
COC = \frac{\ln(\|e_{n+1}\|/\|e_n\|)}{\ln(\|e_n\|/\|e_{n-1}\|)},
\]

where \( e_{n-1} \), \( e_n \) and \( e_{n+1} \) are three consecutive iterations near the root \( \alpha \) and \( e_n = x_n - \alpha \).

**Definition 2.3.** (see [10]) The efficiency index (EI), is defined as \( p^m \), where \( p \) is the rate of convergence of the method and \( m \) is the total number of functions and derivative evaluations at each step of iteration.

**Remark:** If the value of the EI is larger, then the method is more efficient.

2.1 The main results

**Theorem 2.1.** Let \( W \in \mathbb{R}^N \), be a zero of \( F(X) \) and \( e_n = X_n - W \). Then the algorithm defined by (4) has a minimum order of convergence equal to three, and it the satisfies the following error equation

\[
[F'(Y_n) + F'(X_n)]e_{n+1} = \left( -\frac{F(X_n)}{3} + \frac{F''(X_n)}{2} + F'(X_n) - \frac{F''(X_n)}{6} \right) e_n^3 + O(\|e_n\|^4).
\]

**Proof:** From Eq. (4), we have:

\[
F'(Y_n) + F'(X_n)(e_{n+1} - e_n) = -[F'(Y_n) + F'(X_n)]F(X_n)F'(X_n)]^{-1} - 2F'(Y_n),
\]
By using Taylor’s formula, we have
\[
[F'(X_n)]^{-1} F(X_n) = e_n - \frac{[F'(X_n)]^{-1}}{2} F''(X_n)e_n^2 + \frac{[F'(X_n)]^{-1}}{3!} F'''(X_n)e_n^3 + O(\|e_n\|). \tag{5}
\]

Now, applying Taylor’s formula for \( F(Y_n) \) and \( F'(Y_n) \) at the point \( X_n \), we have:
\[
F(Y_n) = F(X_n) - F'(X_n)([F'(X_n)]^{-1} F(X_n)) + \frac{F''(X_n)}{2}([F'(X_n)]^{-1} F(X_n))^2 + ..., \tag{6}
\]

From (5) and (6), we have
\[
F(Y_n) = F(X_n) - e_n F''(X_n) + \frac{1}{2} F''(X_n)[F'(X_n)]^{-1} F'(X_n) + \frac{1}{12} F'''(X_n)e_n^2 + G,
\]
where
\[
G = -\frac{1}{6} F''(X_n)[F'(X_n)]^{-1} F'''(X_n)e_n^3 - \frac{1}{4} e_n F'''(X_n)[F'(X_n)]^{-1} F'''(X_n)e_n^3.
\]

Thus, from above equations, we have
\[
[F'(Y_n) + F'(X_n)] e_{n+1} = (-\frac{1}{3} F''(X_n) + \frac{1}{2} F''(X_n)[F'(X_n)]^{-1} F'(X_n)
\]
\[-\frac{1}{6} F'''(X_n)F'''(X_n)e_n^3 + O(\|e_n\|). \tag{8}
\]

Eq. (8) illustrate that iterative method (4) has cubic convergence.

Now, as a direct consequence, some properties of our scheme are given as follows:

**Proposition 2.1.** The iterative algorithm (4) requires \( 2N \) evaluations of the function and \( 2N^2 \) evaluations of first derivatives.

**Proof.** Consider the N-dimensional case. Hence the Algorithm (4) requires \( 2N \) functional evaluations \( F(X_n) \) and \( F(Y_n) \) and also \( 2N^2 \) of its derivatives \( F'(X_n) \) and \( F'(Y_n) \).

**Proposition 2.2.** The efficiency index of the method (4), is \( 3^{2N+2N^2} \), for \( N \geq 2 \). which is better than the efficiency index of Newton-Simpson’s method (namely that: \( 3^{N+3N^2} \), and also the efficiency index of open Newton’s method (\( 3^{N+3N^2} \)).

**Proof.** For \( N \geq 2 \), we have \( 3^{2N+2N^2} \geq 3^{N+3N^2} \), whence the efficiency index of the method (4), is better than of Newton-Simpson’s method (CT) and Open Newton’s method.
3. Numerical examples

Now, we employ the new method by (4) and compare method with the Newton method (NM), the Newton-Simpson’s method (CT). We carry out computations by using the Matlab and the following stopping criterion is used for computer program

$$\left\| X_{n+1} - X_n \right\|_{\infty} + \left\| F(X_n) \right\|_{\infty} \leq 10^{-10}.$$ 

Case(1): Consider the following problems[2]

Example 3.1 \[ F(x_1, x_2) = (e^{x_1^2} + 8x_1 \sin(x_2)x_1 + x_2 - 1), \]

Example 3.2 \[ F(x_1, x_2) = (x_1^2 - 2x_1 - x_2 + 0.5, x_1^2 + x_2^2 + 4), \]

Example 3.3 \[ F(x_1, x_2) = (e^{x_1^2} - e^{x_2^2}, x_1 - x_2), \]

Example 3.4 \[ F(x_1, x_2, x_3) = (x_1^2 + x_2^2 + x_3^2 - 1, 2x_1^2 - 4x_1, 3x_2^2 - 4x_2 + x_3^2). \]

Table 1. Numerical results and comparison with other methods.

<table>
<thead>
<tr>
<th>Method</th>
<th>Initial</th>
<th>Iterative</th>
<th>Approximation</th>
<th>COC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 3.1</td>
<td>NM (0.2,0.8)</td>
<td>5</td>
<td>(-0.140285,0.140285)</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>CT (0.2,0.8)</td>
<td>4</td>
<td>(-0.140285,0.140285)</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>Algorithm (0.2,0.8)</td>
<td>4</td>
<td>(-0.140285,0.140285)</td>
<td>3.0</td>
</tr>
<tr>
<td>Example 3.2</td>
<td>NM (0.5,0.5)</td>
<td>7</td>
<td>(-0.222214,0.993808)</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>CT (0.5,0.5)</td>
<td>5</td>
<td>(-0.222214,0.993808)</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>Algorithm (0.5,0.5)</td>
<td>5</td>
<td>(-0.222214,0.993808)</td>
<td>3.2</td>
</tr>
<tr>
<td>Example 3.3</td>
<td>NM (0.8,0.8)</td>
<td>4</td>
<td>(0,0)</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>CT (0.8,0.8)</td>
<td>3</td>
<td>(0,0)</td>
<td>4.5</td>
</tr>
<tr>
<td></td>
<td>Algorithm (0.8,0.8)</td>
<td>3</td>
<td>(0,0)</td>
<td>4.4</td>
</tr>
<tr>
<td>Example 3.4</td>
<td>NM (0.5,0.5,0.5)</td>
<td>6</td>
<td>(0.698288,0.628524,0.342564)</td>
<td>2.0</td>
</tr>
<tr>
<td></td>
<td>CT (0.5,0.5,0.5)</td>
<td>4</td>
<td>(0.698288,0.628524,0.342564)</td>
<td>3.0</td>
</tr>
<tr>
<td></td>
<td>Algorithm (0.5,0.5,0.5)</td>
<td>4</td>
<td>(0.698288,0.628524,0.342564)</td>
<td>3.1</td>
</tr>
</tbody>
</table>

The results presented in Table (1) show that our method can be considered as an alternative to Newton-Simpson’s method (CT) and Newton method (NM).

Case(2): Consider the following problems [11]

Example 3.5 \[ F(x) = x^2 + x - 2, \]

Example 3.6 \[ F(x) = x^3 - e^{-x}. \]
Table 2. Numerical results and comparison with He’s method [11]

<table>
<thead>
<tr>
<th>Method</th>
<th>Initial</th>
<th>Iterative</th>
<th>Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 3.5</td>
<td>He</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Algorithm</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>Example 3.6</td>
<td>He</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Algorithm</td>
<td>0</td>
<td>1</td>
</tr>
</tbody>
</table>

The results presented in Table 2 show that our method is in high agreement with He’s approach [11].

4. Conclusion

In this paper, we introduced a cubic convergence iterative method to solve systems of nonlinear equations. This method has simple implementation. As test examples show there is a fast convergence to receive the solution with a required accuracy.

References

Local fractional calculus and fractional complex transform

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Abstract

Based on a recent counter-example of an application of the fractional complex transform, in this paper we proposed the fractional complex transform via local fractional derivative and its demerit can be completely eliminated. Three examples of applications to differential equations in mathematical physics are investigated by the fractional complex transform.

Keywords: Fractional complex transform, local fractional Laplace equation, local fractional wave equation, local fractional heat equation

1. Introduction

The fractional complex transform was first proposed in 2010 [1] to convert fractional differential equations into ordinary differential equations. In previous applications [2-5], the modification of Riemann-Liouville fractional derivatives [6, 7] was adopted, however, a counter-example was found [8], making the method much skeptical. The main problem for its applications is how to define the fractional derivative. This paper finds that the previous demerit can be completely eliminated when the local fractional derivative [9-12] is used.

2. Local fractional calculus

The basic properties of the local fractional derivative are systematically studied in [9-12]. Suppose that \( f(x) \) is local fractional continuous [10-12], the local fractional derivative of \( f(x) \) of order \( \alpha \) at \( x = x_0 \) is given by [9-13]
\[ D_x^{(\alpha)} f(x_0) = f^{(\alpha)}(x_0) = \frac{d^{\alpha} f(x)}{dx^{\alpha}} \bigg|_{x = x_0} = \lim_{x \to x_0} \frac{\Delta^{\alpha} f(x) - f(x_0)}{(x-x_0)^\alpha}, \]  

(1)

where \( \Delta^{\alpha} f(x) = \Gamma(1+\alpha) \Delta f(x) \).

As its inverse operator, local fractional integral is defined as [9-13]

\[ J_x^{(\alpha)} f(x) = \frac{1}{\Gamma(1+\alpha)} \int_a^x f(t)(dt)^\alpha = \frac{1}{\Gamma(1+\alpha)} \lim_{\Delta \to 0} \sum_{j=0}^{N-1} f(t_j)(\Delta t)^\alpha, \]

(2)

with a partition of the interval \([a, b]\), which is defined through \( (t_j, t_{j+1}) \) with \( \Delta x = t_{j+1} - t_j \), \( \Delta t = \max\{\Delta x, \Delta t, \Delta t, ..., \} \), \( j = 0, ..., N-1 \), \( t_0 = a \) and \( t_N = b \). For more results for local fractional order calculus theory, see [9-11].

The chain rule for the local fractional derivative is given as follows [11-13]:

\[ \frac{d^{\alpha} f(x)}{dx^{\alpha}} = f^{(1)}(g) g^{(\alpha)}(x), \]

(3)

where there exist \( f^{(1)}(g) \) and \( g^{(\alpha)}(x) \).

The conditions imposed to Eq. (3) are [10]

\[ |g(x) - g(x_0)| < \varepsilon^\alpha, \]

(4)

and

\[ |f(g) - f(g_0)| < \eta, \]

(5)

which refer to the classical continuity, where \( |x-x_0| < \delta \), for \( \varepsilon, \delta > 0 \) and \( \varepsilon, \delta, \eta \in \mathbb{R} \).

The inequality expressed by Eq. (4) means that [10, 13]

\[ |g(x) - g(x_0)| \leq \tau^\alpha |x-x_0|^{\alpha}, \]

(6)

Hence, we have

\[ \rho^\alpha |x-x_0|^{\alpha} \leq |g(x) - g(x_0)| \leq \tau^\alpha |x-x_0|^{\alpha}. \]

(7)

In Eq. (7), \( g(x) \) presents a bi-Lipschitz mapping [9, 13] of a fractal dimension \( \alpha \) because it can be measured by a fractal measure [9], namely

\[ H^\alpha (F \cap (x, x_0)) = (x-x_0)^\alpha. \]

(8)

Therefore, it follows that the classical continuity can be derived from the Lipschitz mapping, while the local fractional continuity is related to the bi-Lipschitz mapping because the local fractional continuity was introduced. Thus, \( f(g) \) is a Lipschitz mapping as a variable \( g \), while \( g(x) \) is a bi-Lipschitz mapping for a variable \( x \). A comparison between the local modified Riemann-Liouville fractional derivative [6, 7] and the local fractional classical derivative [13, 14] reveals that the operating functions are Lipschitz mappings.
2. Fractional complex transform

The fractional complex transform met some problems in applications when the modification of Riemann-Liouville fractional derivatives was adopted due to the complex chain rule. The chain rule for local fractional calculus is simple as given in Eq.(3). So the fractional complex transform becomes valid for local fractional differential equations.

Example 1

Consider the local fractional differential equation in the form

\[
\frac{d^\alpha U_1(x)}{dx^\alpha} + \frac{d^\alpha U_2(y)}{dy^\alpha} = 0, \quad 0 < \alpha \leq 1
\]  

(9)

By the fractional complex transform \[2, 3\]

\[
\begin{align*}
X &= \left(\frac{px}{\Gamma(1+\alpha)}\right)^{\alpha}, \\
Y &= \left(\frac{qy}{\Gamma(1+\alpha)}\right)^{\alpha},
\end{align*}
\]  

(10)

where \(p\) and \(q\) are constants.

Eq.(9) turns out to be the following ordinary differential equation

\[
p^{\alpha} \frac{dU_1(X)}{dX} + q^{\alpha} \frac{dU_2(Y)}{dY} = 0
\]  

(11)

Example 2

Let us consider the following partial differential equation

\[
\frac{\partial^\alpha U_1(x, y)}{\partial x^\alpha} + \frac{\partial^\alpha U_2(x, y)}{\partial y^\alpha} = 0
\]  

(12)

By the fractional complex transform (10) and the chain rule for local fractional calculus, we have

\[
\frac{\partial^\alpha U_1(x, y)}{\partial x^\alpha} = \frac{\partial U_1(X, Y)}{\partial X} \frac{\partial^\alpha X}{\partial x^\alpha} + \frac{\partial U_1(X, Y)}{\partial Y} \frac{\partial^\alpha Y}{\partial x^\alpha} = p^{\alpha} \frac{\partial U_1(X, Y)}{\partial X}
\]  

(13)

and

\[
\frac{\partial^\alpha U_2(x, y)}{\partial y^\alpha} = \frac{\partial U_2(X, Y)}{\partial Y} \frac{\partial^\alpha Y}{\partial y^\alpha} + \frac{\partial U_2(X, Y)}{\partial X} \frac{\partial^\alpha X}{\partial y^\alpha} = q^{\alpha} \frac{\partial U_2(X, Y)}{\partial Y}.
\]  

(13)

Finally Eq. (12) becomes the following partial differential equation
\[
p^a \frac{\partial U_1(X,Y)}{\partial X} + q^a \frac{\partial U_2(X,Y)}{\partial Y} = 0. \tag{14}
\]

**Example 3**

The last equation to start with is
\[
\frac{\partial^2 U_2(x,y)}{\partial x^{2a}} + \frac{\partial^2 U_1(x,y)}{\partial y^{2a}} + \frac{\partial^2 U_1(x,y)}{\partial x^{2a}} + \frac{\partial^2 U_2(x,y)}{\partial y^{2a}} = 0. \tag{15}
\]

Similarly by the fractional complex transform (10), we have
\[
\frac{\partial^2 U_1(x,y)}{\partial x^{2a}} = \frac{\partial^2 U_1(X,Y)}{\partial X^2} \frac{\partial^alpha}{\partial x^\alpha} + \frac{\partial^2 U_1(X,Y)}{\partial Y^2} \frac{\partial^alpha}{\partial y^\alpha} + \frac{\partial^2 U_2(X,Y)}{\partial x^{2a}} = p^{2a} \frac{\partial^2 U_1(X,Y)}{\partial X^2}. \tag{16}
\]

\[
\frac{\partial^2 U_2(x,y)}{\partial y^{2a}} = q^{2a} \frac{\partial^2 U_2(X,Y)}{\partial Y^2} \frac{\partial^alpha}{\partial y^\alpha} + q^a \frac{\partial^2 U_2(X,Y)}{\partial x^2} \frac{\partial^alpha}{\partial x^\alpha} = q^{2a} \frac{\partial^2 U_2(X,Y)}{\partial X^2}. \tag{17}
\]

\[
\frac{\partial^2 U_1(x,y)}{\partial y^{2a}} = p^a \frac{\partial^2 U_1(X,Y)}{\partial x^2} \frac{\partial^alpha}{\partial y^\alpha} + p^a \frac{\partial^2 U_1(X,Y)}{\partial x^2} \frac{\partial^alpha}{\partial x^\alpha} = q^a \frac{\partial^2 U_1(X,Y)}{\partial Y^2}. \tag{18}
\]

\[
\frac{\partial^2 U_2(x,y)}{\partial y^{2a}} = q^a \frac{\partial^2 U_2(X,Y)}{\partial Y^2} \frac{\partial^alpha}{\partial y^\alpha} + q^a \frac{\partial^2 U_2(X,Y)}{\partial x^2} \frac{\partial^alpha}{\partial x^\alpha} = q^a \frac{\partial^2 U_2(X,Y)}{\partial Y^2}. \tag{19}
\]

Using the above relations, Eqs.(16)-(19), we convert Eq.(15) into the following one
\[
p^{2a} \frac{\partial^2 U_2(X,Y)}{\partial X^2} + p^a q^a \frac{\partial^2 U_1(X,Y)}{\partial X^2} + q^a p^a \frac{\partial^2 U_2(X,Y)}{\partial Y^2} + q^{2a} \frac{\partial^2 U_2(X,Y)}{\partial Y^2} = 0. \tag{20}
\]

**3 Applications to differential equations in mathematical physics**

In the section, we investigate the classical Laplace, the classical heat and the classical wave equations respectively [16]:

\[
\frac{\partial^2 U(X,Y)}{\partial X^2} + \frac{\partial^2 U(X,Y)}{\partial Y^2} = 0 \quad \text{(Laplace equation)}, \tag{21}
\]

\[
\frac{\partial U(X,T)}{\partial T^a} \cdot \frac{\partial^2 U(X,T)}{\partial X^2} = 0 \quad \text{(Heat equation)}, \tag{22}
\]

\[
\frac{\partial^2 U(X,T)}{\partial T^2} \cdot \frac{\partial^2 U(X,T)}{\partial X^2} = 0 \quad \text{(Wave equation)}. \tag{23}
\]

Suppose that the fractional complex transform is
The fractional complex transform [2, 3] revives again for the local fractional calculus, and it cleans up solution of local fractional differential equations [17-20]. The transform makes the local fractional calculus more accessible for researchers working in various disciplines of science and engineering.

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References


Double-diffusive convection from a permeable vertical surface under convective boundary condition in the presence of heat generation and thermal radiation

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Abstract

We analyze the simultaneous effects of thermal and concentrations on a mixed convection boundary layer flow over a permeable surface under convective surface boundary condition in the presence of heat generation and thermal radiation. Using a similarity variable, the governing nonlinear partial differential equations have been transformed into a set of coupled nonlinear ordinary differential equations, which are solved numerically using Maple 14 which uses a fifth-sixth order Runge–Kutta–Fehlberg algorithm together with shooting method. It is seen that as buoyancy force parameter N and the Biot number Bi increases, the skin-friction coefficient, heat transfer rate and the mass transfer rate also increases. It is also noticed that increasing the thermal radiation and the internal heat generation parameters, it enhances the skin-friction coefficient and the mass transfer rate while it decreases the heat transfer rate at the wall surface. The heat generation parameters γ enhances the skin-friction coefficient and the mass transfer rate. Similarly, the embedded flow parameters have significant influence on the double-diffusive convection and that heat generation and radiation parameters enhances the skin-friction coefficient and mass transfer while it reduces the heat transfer rate.

Keywords: Double-diffusive convection, convective boundary condition, heat generation, thermal radiation.

1. Introduction

In many natural and technological processes, temperature and mass or concentration diffusion act together to create a buoyancy force which drives the fluid and this is known as double-diffusive convection, or combined heat and mass concentration transfer convection. In oceanography, convection processes involve thermal and salinity gradients and this is referred to as thermohaline convection, whilst surface gradients of the temperature and the solute concentration are referred to as
Marangoni convection. The term double-diffusive convection is now widely accepted for all processes which involve simultaneous thermal and concentration (solutal) gradients and provides an explanation for a number of natural phenomena. Because of the coupling between the fluid velocity field and the diffusive (thermal and concentration) fields, double-diffusive convection is more complex than the convective flow which is associated with a single diffusive scalar, and much different behaviour may be expected. Such double-diffusive processes occur in many fields, including chemical engineering (drying, cleaning operations, evaporation, condensation, sublimation, deposition of thin films, energy storage in solar ponds, roll-over in storage tanks containing liquefied natural gas, solution mining of salt caverns for crude oil storage, casting of metal alloys and photosynthesis), solid-state physics (solidification of binary alloy and crystal growth), oceanography (melting and cooling near ice surfaces, sea water intrusion into freshwater lakes and the formation of layered or columnar structures during crystallization of igneous intrusions in the Earth's crust), geophysics (dispersion of solvent materials or particulate matter in flows), etc. A clear understanding of the nature of the interaction between thermal and mass or concentration buoyancy forces are necessary in order to control these processes. Many researchers (see, Raptis et al., 2004, Hayat et al., 2007, Ishak 2010, Aziz, 2009, 2010, Hayat et al., 2007Makinde and Olanrewaju, 2011) have shown interest in recent years for obtaining self-similar solutions of boundary layer flows with thermal and/or mass diffusion. Self-similar solutions for the coupled momentum and thermal boundary layer over flat surfaces are presented in convective heat transfer textbooks namely Bejan (2004) and Incropera et al. (2007). Moreover, different studies (see, Rogers, 1992, Shu and Pop, 1988) have considered variations of temperature and heat flux at the plate related to heat transfer application. Both Aziz, (2009) and Ishak (2010) have reported that a similarity solution is possible if the convective heat transfer of the fluid heating the plate on its surface is proportional to $x^{-1/2}$. In fact, Ishak (2010) has included the effects of suction and injection on flat surface in his investigation. Not surprisingly, the study of similarity solutions for laminar thermal boundary layer flows over flat surfaces or stretching/shrinking sheet under convective surface boundary condition has attracted the attention of many investigators, such as, Magyari (2011), Ishak et al. (2010), Bataller (2008), Makinde and Aziz (2010), Yao et al. (2011), Makinde and Olanrewaju (2010), Makinde (2010) and Olanrewaju et al. (2011).

It may be remarked that earlier studies did not include the effect of mass diffusion. However, in many real boundary layer flows, the flow, and heat transfer and mass diffusion are always coupled. Very recently, Subhashini et al. (2011) examined double-diffusive convection from a permeable vertical surface under convective boundary condition. Reddaiah and Prasada (2012) examined the convective heat and mass transfer flow of a viscous fluid through a porous medium in an duct-by finite element method. Surati and Timl (2012) investigated the natural convection flow of non-Newtonian fluids. Similarly, Reddaiah and Prasada (2012) studied the effect of radiation on Hydromagnetic convective heat transfer flow of a viscous electrically conducting fluid in a non-uniformly heated vertical channel. Ibrahim and Reddy (2012) examined the radiation and mass transfer effects on MHD free convection flow along a stretching surface with viscous dissipation and heat generation. Hayat et al.(2010) studied the radiation and mass transfer effects on magnetohydrodynamic unsteady flow induced by a shrinking sheet and similarly, Hayat and Qasim (2010) examined the influence of thermal radiation and Joule heating on MHD flow of a Maxwell fluid in the presence of thermophoresis.

The objective of this present study is to extend Subhashini et al. (2011) to include heat generation and thermal radiation which has applications in industry, science and engineering. According to the author's knowledge, no one has worked on this present problem. The self-similar solution of the coupled non-linear partial differential equations governing the mixed convective flow has been solved numerically using Runge-Kutta of order sixth alongside with shooting technique. These results for some particular cases are compared with those of Olanrewaju and Makinde (2010), Subhashini et al. (2011) and Olanrewaju et al(2011).
Makinde and Olanrewaju (2011) and Olanrewaju, et al. (2011) only studied a single diffusive scalar and much behavior may occur in many fields. In engineering applications, double-diffusive processes are more needed than a single diffusive scalar. Similarly, a vertical permeable surface is used to gain more insight into the practical aspect of the research work compared to the recent work of Makinde and Olanrewaju (2011) and Olanrewaju et al. (2011). Finally, the present work considered a steady two dimensional laminar mixed convection flow along a static permeable flat surface in a viscous fluid, where \( V_w(x) < 0 \) corresponds to suction and \( V_w(x) > 0 \) corresponds to blowing or injection. The method used is similar to Makinde and Olanrewaju (2011) except that it was implemented in Maple 15 for better accuracy.

2. Problem formulation

We consider a steady two dimensional laminar mixed convection flow along a static permeable flat surface in a viscous fluid of temperature \( T_\infty \) and concentration \( C_\infty \). It is assumed that the free stream moves with a constant velocity \( U_\infty \) (see Fig. 1). It is also assumed that the left side of the plate is heated or cooled by convection \( T_f \), where \( T_f > T_\infty \) corresponds to a heated plate (assisting flow) and \( T_f < T_\infty \) corresponds to a cooled surface (opposing flow), respectively. The buoyancy forces arise due to the variations in temperature and concentration of fluid. The Boussinesq approximation is invoked for the fluid properties to relate the density changes to temperature and concentration and to couple in this way the temperature and concentration fields to the flow field.

\[
-k \frac{\partial T}{\partial y} = h_f [T_f - T], \quad Q
\]

\[C = C_w, \quad v = V_w(x)\]

\[u = 0\]

\[y\]

**Fig. 1.** Flow configuration and coordinate system

Under the assumptions, the governing boundary equations can be written as Subhashini et al. (2011)

\[
\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (1)
\]

\[
u \frac{\partial u}{\partial x} + \nu \frac{\partial u}{\partial y} = \nu \frac{\partial^2 u}{\partial y^2} + g\beta (T - T_\infty) + g\beta^* (C - C_\infty), \quad (2)
\]

\[
u \frac{\partial T}{\partial x} + \nu \frac{\partial T}{\partial y} = \alpha \frac{\partial^2 T}{\partial y^2} - \frac{\alpha}{k} \frac{\partial q_r}{\partial y} + \frac{Q}{\rho c_p} (T - T_\infty) \quad (3)
\]
Along with the relevant boundary conditions

\[ u = 0, \quad v = V_w(x), \quad C = C_w \quad \text{at} \quad y = 0, \]

\[ u \to U_\infty, \quad C \to C_\infty \quad \text{as} \quad y \to \infty, \]

where \( V_w(x) < 0 \) corresponds to suction and \( V_w(x) > 0 \) corresponds to blowing or injection, respectively. On the other hand, it is further assumed that the fluid temperature \( T_f \) which provides a heat transfer coefficient \( h_f \). Under this assumption, the boundary conditions for the thermal field may be written as Olanrewaju et al. (2011)

\[ -k \frac{\partial T}{\partial y} = h_f (T_f - T(x,0)) \quad \text{at} \quad y = 0, \]

\[ T \to T_\infty \quad \text{as} \quad y \to \infty. \]

where \( u \) and \( v \) are the \( x \) (along the plate) and the \( y \) (normal to the plate) components of the velocity vectors, respectively, \( T \) is the temperature, \( \nu \) is the kinematics viscosity of the fluid, and \( \alpha \) is the thermal diffusivity of the fluid and \( \beta \) is the thermal expansion coefficient, \( \beta^* \) is the solutal expansion coefficient, \( Q \) is the heat release per unit per mass, \( g \) is the gravitational acceleration, \( q_r \) is the radiative heat flux, \( k \) is the thermal conductivity, \( D \) is the coefficient of mass diffusivity, respectively.

The radiative heat flux \( q_r \) is described by Roseland approximation such that

\[ q_r = -\frac{4\sigma^* \partial T^4}{3K \partial y}, \]

where \( \sigma^* \) and \( K \) are the Stefan-Boltzmann constant and the mean absorption coefficient, respectively. Following Olanrewaju et al. (2011), we assume that the temperature differences within the flow are sufficiently small so that the \( T^4 \) can be expressed as a linear function after using Taylor series to expand \( T^4 \) about the free stream temperature \( T_\infty \) and neglecting higher-order terms. This result is the following approximation:

\[ T^4 \approx 4T_\infty^3 T - 3T_\infty^4. \]

Using (7) and (8) in (3), we obtain

\[ \frac{\partial q_r}{\partial y} = -\frac{16\sigma^* \partial^2 T^4}{3K \partial y^2}. \]

Introducing a similarity variable \( \eta \) and a dimensionless stream function \( f(\eta) \), temperature \( \theta(\eta) \) and \( \phi(\eta) \) as

\[ \eta = y \sqrt{\frac{U_\infty}{U_x}}, \quad u = f' + \frac{1}{2} \left( \frac{U_x}{x} \right) (\eta f' - f), \quad \theta = \frac{T - T_\infty}{T_f - T_\infty}, \quad \phi = \frac{C - C_\infty}{C_w - C_\infty}, \]

where prime symbol denotes differentiation with respect to \( \eta \). In order that similarity solutions of Eqs. (1) – (7) exist, we take (see, Olanrewaju, et al. (2011))
Where \( c, f_w \) and \( Q_0 \) are constants with \( f_w > 0 \) for the transpiration (suction) rate at the surface, \( f_w < 0 \) for injection and \( f_w = 0 \) for an impermeable surface. On substituting the new variables (10) into Eqs. (2), (3) and (4) we obtained the following set of ordinary differential equations

\[
f'''' + \frac{1}{2} f'''' + \lambda (\theta + N\phi) = 0, \tag{12}
\]

\[
\left[ 1 + \frac{4}{3} Ra \right] \theta'' + \frac{1}{2} Pr f\theta' + Pr \gamma \theta = 0, \tag{13}
\]

\[
\phi'' + \frac{1}{2} Sc f \phi' = 0, \tag{14}
\]

with the boundary conditions

\[
f(0) = f_w, \quad f'(0) = 0, \quad \theta'(0) = -B\left[1 - \theta(0)\right], \quad \phi(0) = 1,
\]

\[
f''(\infty) = 1, \quad \theta(\infty) = 0, \quad \phi(\infty) = 0, \tag{15}
\]

where primes denote differentiation with respect to \( \eta \) and the eight parameters appearing in Eqs. (12-15) are defined as follows,

\[
Pr = \frac{\nu}{\alpha}, \quad Bi = \frac{c (\nu / U_x)}{k}, \quad Ra = \frac{4 \alpha \sigma T^3}{K}, \quad Sc = \frac{\nu}{D},
\]

\[
\gamma = \frac{Q_0}{\rho c_p U_x}, \quad N = \frac{Gr^*}{Gr_x}, \quad \lambda = \frac{Gr_x}{Re_x^2}, \quad Gr_x = \frac{g \beta (T_f - T_x) x^3}{\nu^2}, \tag{16}
\]

\[
Gr^*_x = \frac{g \beta^*(C_w - C_x) x^3}{\nu^2}, \quad Re_x = \frac{U_x x}{\nu},
\]

where \( Pr, Bi, Ra, Sc, \lambda, \gamma, Sc \) and \( N \) represents the Prandtl number, Biot number, thermal radiation parameter, mixed convection parameter, heat generation parameter, Schmitz number and the buoyancy force parameter, respectively. Note that \( Gr_x \) and \( Gr^*_x \) are the local Grashof numbers while \( Re_x \) is the local Reynolds number. It is evident that the numerical solution is valid as long as it represents local similarity solution since the parameters still depend on an independent variable \( x \). Similarly \( \lambda < 0 \) corresponds to opposing flow while \( \lambda > 0 \) corresponds to assisting flow.

Other physical quantities of interest in this problem, namely; the skin friction parameter \( (\tau) \) and the Nusselt number \( (Nu) \) and Sherwood number \( (Sh) \) can be easily computed. These quantities are defined in dimensionless terms as: \( \tau \equiv -F''(0), \quad Nu \equiv \theta'(0) \) and \( Sh \equiv \phi'(0) \).

### 3. Numerical procedure

The set of non-linear ordinary differential equations (12)–(14) with boundary conditions in (15) have been solved numerically by using the Runge–Kutta integration scheme with a modified version of the Newton–Raphson shooting method with \( Pr, Bi, Ra, Sc, \lambda, \gamma, f_w \) and \( N \) as prescribed parameters. The computations were done by a program which uses a symbolic and computational computer.
language MAPLE (Heck, 2003) A step size of $\Delta \eta = 0.001$ was selected to be satisfactory for a convergence criterion of $10^{-7}$ in nearly all cases. The value of $\eta_c$ was found to each iteration loop by the assignment statement $\eta_c = \eta_c + \Delta \eta$. The maximum value of $\eta_c$, to each group of parameters Pr, Bi, Ra, Sc, $\lambda$, $\gamma$, $f_w$ and N is determined when the values of unknown boundary conditions at $\eta = 0$ not change to successful loop with error less than $10^{-7}$.

4. Results and discussion

In order to get a clear insight of the physical problem, the velocity, temperature and concentration have been discussed by assigning numerical values to the parameters encountered in the problem. To be realistic, the values of Schmidt number ($Sc$) are chosen for hydrogen ($Sc = 0.22$), water vapor ($Sc = 0.62$), ammonia ($Sc = 0.78$) and Propyl Benzene ($Sc = 2.62$) at temperature 25°C and one atmospheric pressure. The values of Prandtl number is chosen to be $Pr = 0.7$ which represents air at temperature 25°C and one atmospheric pressure. It is seen from Tables 1 and 2 that there is a perfect agreement with the previously published articles. From Table 3, it is seen that as buoyancy force parameter N and the Biot number Bi increases, the skin-friction coefficient, heat transfer rate and the mass transfer rate increases. It is also noticed that increasing the thermal radiation and the internal heat generation parameters, it enhances the skin-friction coefficient and the mass transfer rate while it decreases the heat transfer rate at the wall surface.

**Table 1** Comparison of $f''(0)$ and $-\theta'(0)$ for various values of a, $\lambda$ and Pr when $f_w = 0$, $N = 0$, $Ra = 0$, $Sc = 0$ and $\gamma = 0$ with those of Makinde and Olanrewaju [17] and Subhashini et al. [20].

<table>
<thead>
<tr>
<th>Bi</th>
<th>$\bar{e}$</th>
<th>Pr</th>
<th>$f''(0)$ Makinde &amp; Olanrewaju [17]</th>
<th>$-\theta'(0)$ Makinde &amp; Olanrewaju [17]</th>
<th>$f''(0)$ Subhashini et al. [20]</th>
<th>$-\theta'(0)$ Subhashini et al. [20]</th>
<th>Present results $f''(0)$</th>
<th>Present results $-\theta'(0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.1</td>
<td>0.72</td>
<td>0.36881</td>
<td>0.07507</td>
<td>0.36875</td>
<td>0.07505</td>
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</table>

**Table 2** Comparison of $f''(0)$ and $-\theta'(0)$ for various values of a, $\lambda$, Pr, $\gamma$ and Ra when $f_w = 0$, $N = 0$, $Sc = 0$ with those of Olanrewaju et al. [19]

<table>
<thead>
<tr>
<th>Bi</th>
<th>$\bar{e}$</th>
<th>Pr</th>
<th>$\bar{a}$</th>
<th>Ra</th>
<th>$f''(0)$ Olanrewaju et al. [19]</th>
<th>$-\theta'(0)$ Olanrewaju et al. [19]</th>
<th>Present result $f''(0)$</th>
<th>Present result $-\theta'(0)$</th>
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</table>
Table 3 Computations showing $f''(0)$, $-\theta'(0)$ and $\phi'(0)$ for various values of $N$, $fw$, $Bi$, $Ra$ and $\gamma$ when $\lambda = 1$, $Pr = 0.7$ and $Sc = 0.94$.

<table>
<thead>
<tr>
<th>N</th>
<th>$fw$</th>
<th>Bi</th>
<th>Ra</th>
<th>$\alpha$</th>
<th>$f''(0)$</th>
<th>$-\theta'(0)$</th>
<th>$\phi'(0)$</th>
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The variations of buoyancy parameter $\lambda$ on the velocity, temperature and concentration profiles for fixed values of other controlling flow parameters are displayed in Figures 2-4. In the presence of assisting buoyancy effect, the buoyancy force shows a significant overshoot in the velocity profiles near the wall which resulted in the thickening of the velocity boundary layer thickness but for opposing flow, the velocity profiles reduces resulted in the thinning of the velocity boundary layer thickness. For the thermal and concentration profiles, the assisting flow reduces the thermal and concentration boundary layer thickness while the opposing flow enhances the thermal and the concentration thermal boundary layer thickness. Effects of Prandtl number on temperature profile is depicted in Figure 5. It is evident that increasing Prandtl number reduces the thermal boundary layer thickness as expected. Figures 6, 7 and 8 depict the effects of Biot number, radiation parameter and the internal heat generation on the thermal boundary. It shows that increasing these flow parameters enhances the thermal boundary layer thickness as expected. The influence of suction/injection on the velocity, temperature and the concentration profile were displayed in Figures 9-11. As it was expected that suction step down flow as a result of that increases the temperature of the fluid while injection behaved other way round. Similarly increasing the suction parameter reduces the concentration boundary layer thickness. The effect of Schmidt number was displayed in Figure 12 and it was established that increasing this so called parameter decreases the concentration boundary layer thickness. Finally, Fig. 13 displays the effect of N (ratio of concentration buoyancy force to thermal buoyancy force parameters) on the velocity profile for various values of $\lambda$ when $Pr=0.7$, $fw=1$, $Bi=1$, $Ra = 0.1$, $\gamma = 0.1$ and $Sc=0.94$. The positive values of N imply that both buoyancy forces act in the same direction. On the contrary, the negative values of N appear when thermal and concentration buoyancy forces act in opposite directions. As N increases, the velocity overshoot increases for buoyancy assisting flow ($\lambda > 0$) and the reverse flow region increases for buoyancy opposing flow ($\lambda < 0$), near the wall. The physical reason is that the buoyancy forces act like pressure gradient which accelerates/decelerates the fluid within the boundary layer. The effect of N on the temperature and concentration profiles is highly insignificant on the temperature and concentration profiles because the parameter N is explicitly present only in the momentum equation and those profiles are not shown for the sake of brevity.
Figure 2: Effects of $\lambda$ on the velocity distribution for fixed values of $Sc = 0.94$, $Ra = 0.1$, $\gamma = 0.1$, $Pr = 0.7$, $N = 0.5$, $Bi = 1$ and $fw = 1$.

Figure 3: Effects of $\lambda$ on the temperature distribution for fixed values of $Sc = 0.94$, $Ra = 0.1$, $\gamma = 0.1$, $Pr = 0.7$, $N = 0.5$, $Bi = 1$ and $fw = 1$. 
Figure 4: Effects of $\lambda$ on the concentration distribution for fixed values of $Sc = 0.94$, $Ra = 0.1$, $\gamma = 0.1$, $Pr = 0.7$, $N = 0.5$, $Bi = 1$ and $fw = 1$.

Figure 5: Effects of $Pr$ on the temperature distribution for fixed values of $Sc = 0.94$, $Ra = 0.1$, $\gamma = 0.1$, $\lambda = 0.1$, $N = 0.5$, $Bi = 1$ and $fw = 1$. 
Figure 6: Effects of Bi on the temperature distribution for fixed values of $Sc = 0.94$, $Ra = 0.1$, $\gamma = 0.1$, $\lambda = 0.1$, $N = 0.5$, $Pr = 0.7$ and $fw = 1$.

Figure 7: Effects of Ra on the temperature distribution for fixed values of $Sc = 0.94$, $Bi = 1$, $\gamma = 0.1$, $\lambda = 0.1$, $N = 0.5$, $Pr = 0.7$ and $fw = 1$. 
Figure 8: Effects of $\beta$ on the temperature distribution for fixed values of $\text{Sc} = 0.94$, $\text{Bi} = 1$, $\text{Ra} = 0.1$, $\lambda = 0.1$, $N = 0.5$, $\text{Pr} = 0.7$ and $fw = 1$.

Figure 9: Effects of $fw$ on the velocity distribution for fixed values of $\text{Sc} = 0.94$, $\text{Bi} = 1$, $\text{Ra} = 0.1$, $\lambda = 0.1$, $N = 0.5$, $\text{Pr} = 0.7$ and $\gamma = 0.1$. 
Figure 10: Effects of \( fw \) on the temperature distribution for fixed values of \( Sc = 0.94, Bi = 1, Ra = 0.1, \lambda = 0.1, N = 0.5, Pr = 0.7 \) and \( \gamma = 0.1 \).

Figure 11: Effects of \( fw \) on the concentration distribution for fixed values of \( Sc = 0.94, Bi = 1, Ra = 0.1, \lambda = 0.1, N = 0.5, Pr = 0.7 \) and \( \gamma = 0.1 \).
Figure 12: Effects of Sc on the concentration distribution for fixed values of $fw = 1$, $Bi = 1$, $Ra = 0.1$, $\lambda = 0.1$, $N = 0.5$, $Pr = 0.7$ and $\gamma = 0.1$.

Figure 13: Effects of N on the velocity distribution for fixed values of $fw = 1$, $Bi = 1$, $Ra = 0.1$, $\lambda = 0.1$, $Sc = 0.94$, $Pr = 0.7$ and $\gamma = 0.1$. 
5. Conclusion

Numerical solutions are obtained for double-diffusive convection from a permeable vertical surface under convective boundary condition in the presence of heat generation and thermal radiation. The numerical solution is valid as long as it represents local similarity solution since the parameters still depends on an independent variable x. Using a similarity variable, the governing nonlinear partial differential equations have been transformed into a set of coupled nonlinear ordinary differential equations, which are solved numerically using Maple 14 which uses a fifth-sixth order Runge-Kutta-Fehlberg algorithm together with shooting method. Conclusions of the study are as follows:

- The radiation Ra and the heat generation parameters $\gamma$ enhances the skin-friction coefficient and the mass transfer rate while it reduces the heat transfer rate at the wall as we increase radiation and heat generation parameters.
- Increasing the Biot number Bi and the buoyancy force parameter N enhances the skin-friction coefficient, heat transfer rate $Nu$ and the mass transfer rate $Sh$ at the wall plate.
- Increasing the values of N, the velocity overshoots increases for buoyancy assisting flow.
- Heat and mass transfer rate is been affected due to injection/suction parameters.
- Similarly, higher Prandtl number causes thinner thermal boundary layer and higher Schmidt number causes thinner concentration boundary layer.

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References

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